266. Dynamic stability of a flexible rod under parametric excitation

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Abstract. Vertically mounted and cantilevered homogeneous rod with a constant cross section is considered. At the unstrained state, the rod's axis is straight. It is supposed that lateral stiffness is such that the rod buckles in static conditions only under gravity and its axis bends. Flexural oscillations of the rod are investigated when its base vibrates vertically. Stabilization circumstances of the rectilinearity of the rod's axis are determined depending on excitation parameters, internal and external damping, and rod flexibility.

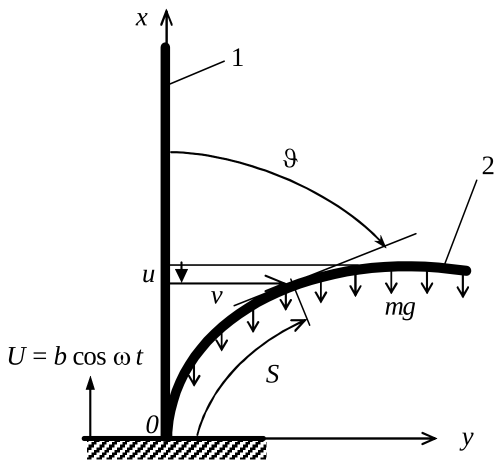
Keywords: flexible rod, parametric excitation, stabilization, parametric resonance, dynamical stability.

Introduction

The rectilinear homogeneous elastic cantilevered rod with a constant cross section is considered. At the unstrained state, the rod's axis coincides with vertical axis x of the rectangular coordinate system xOy with origin placed in the rigid support of the rod (Fig. 1). In the absence of external forces the flexible rectilinear rod buckles under uniformly distributed along rod length gravity and its axis bends (Fig. 1) if lateral stiffness parameter $\frac{mgL^3}{EI} > 7.839$ (m - mass of rod length unit, L - length of the rod, EI - flexural stiffness, g - gravitational acceleration) [1].

It is known if the base of the rod subject to vertical oscillation when it is possible to back up (stabilize) origin rectilinearity of the rod's axis due to parametric excitation at certain frequency and amplitude [2, 3]. In other words, the lateral stiffness of such rod increases under parametric excitation.

The purpose of this work is to investigate parametric oscillation of the flexible rod and to determine stabilization conditions of the rectilinear form of the rod's axis under vertical vibration of its base depending on parameters of excitation, internal and external damping, and rod flexibility parameter.



Analytical model

It is considered continual model of the flexible rod taking into account internal and external (relative to absolute space) friction. Behavior of stability domain boundaries and parametric resonances are determined on the basis of numeric simulation for the purpose to obtain obvious and simple results (for next verification).

Differential equation describing behavior of the flexible rod under parametric excitation was derived in our previous paper [2]. At that, it was assumed following:

– the base of the rod oscillates harmonically in vertical direction under the law $U = b\cos\omega t$, where b - amplitude, ω - frequency of oscillations.
– the rod's axis is inextensible;
– finite rotations of sections are taken into account;
– rotary inertia of cross sections isn't considered;
– expressions for axle curvature κ and axial displacement u as functions of deflection v are taken into account relative to geometrical nonlinearity of the third infinitesimal order, i.e.:

$$\kappa = v''\left(1-v'^2\right)^{-1/2} = v''\left(1+v'^2/2+O\left(v'^4\right)\right)\approx$$
$$\approx v''\left(1+v'^2/2\right),\quad u'=1-\sqrt{1-\sin^2 v'}\approx\frac{1}{2}v'^2 \tag{1}$$

– external friction is proportional to absolute velocity $\bar{V}$ of the rod's section with damping coefficient d (linear-viscous damping model):

$$\bar{V}=-\left(b\omega\sin\omega t+\partial u/\partial t\right)\bar{i}+\partial v/\partial t\,\bar{j}\,, \tag{2}$$

where $\left\{\bar{i},\bar{j}\right\}$ – unit vector of the coordinates $\left\{x,y\right\}$;

– Foight's model is applied to take account of internal friction; according to model internal moment

$$M_z\left(t,S\right)=EI\left[\kappa\left(t,S\right)+d_I\,\frac{\partial\kappa\left(t,S\right)}{\partial t}\right], \tag{3}$$

where S - angular position of the rod's section setting from the base, d_I - coefficient of internal friction;

– boundary conditions:

$$S=0:\quad v=0,\quad v'=0;\quad S=L:\quad v''=0,\quad v'''=0$$

Nonlinear equation of the third infinitesimal order relative to displacements in dimensionless form is given by [2]:

$$\ddot{\xi}+2\psi\dot{\xi}+\xi^{IV}+p_\Sigma\left(\tau\right)\left[\left(1-\zeta\right)\xi'\right]'=$$
$$=-2\psi_I\,\dot{\xi}^{IV}-2\psi_I\,\varepsilon\frac{\partial}{\partial\tau}\left[\xi'\left(\xi''\xi'\right)'\right]'+$$
$$+\varepsilon\left\{\begin{aligned}&\xi'\int_\zeta^1 d\zeta_1\int_0^{\zeta_1}\left(\dot{\xi}'^2+\xi'\ddot{\xi}'+2\psi\xi'\dot{\xi}'\right)d\zeta_2-\\&-\xi'\left(\xi''\xi'\right)'-p_\Sigma\left(\tau\right)\left(1-\zeta\right)\xi'^3/2\end{aligned}\right\}', \tag{4}$$

$$p_\Sigma\left(\tau\right)=\gamma-\beta\Omega^2\cos\Omega\tau-2\psi\beta\Omega\sin\Omega\tau \tag{4}$$

where points denote dimensionless time differentiation τ, strokes - dimensionless angular position differentiation ζ; boundary conditions:

$$\zeta=0:\quad \xi=0,\quad \xi'=0;\quad \zeta=1:\quad \xi''=0,\quad \xi'''=0.$$

The following dimensionless parameters and complexes are used here:

$$\tau=\frac{t}{L^2}\sqrt{\frac{EI}{m}},\quad \zeta=\frac{S}{L},\quad \xi=\frac{\nu}{h},\quad \Omega=\omega L^2\sqrt{\frac{m}{EI}},$$

$$\beta=\frac{b}{L},\quad \varepsilon=\left(\frac{h}{L}\right)^2,\quad \gamma=\frac{mgL^3}{EI},\quad \psi_I=\frac{d_I}{2L^2}\sqrt{\frac{EI}{m}}, \tag{5}$$

$$\psi=\frac{dL^2}{2\sqrt{mEI}}<1.$$

Solution of motion equation

Possibility in principle of stabilization of the rectilinear form of the rod's axis in case of parametric excitation for particular values of the system parameters was shown in [2]. At that, single-mode approximation was found with use of Galerkin's technique. In that case, analysis is equivalent to research of Kapitsa's pendulum and results are adduced for nonlinear case of finite oscillations amplitude. In this paper, domains of excitation parameters are determine depending on flexibility parameter and damping (external and internal) that provides stabilization of the rod's vertical axis. With purpose to extend analyzable frequency region it is used multimodal approximation of the solution:

$$\xi\left(\zeta,\tau\right)\approx\sum_{i=1}^{n}q_i\left(\tau\right)\varphi_i\left(\zeta\right), \tag{6}$$

where $q_i\left(\tau\right)$ - amplitude functions, $\varphi_i\left(\zeta\right)$ - coordinate functions.

Let us consider linearized Eq. (4) for determination of stability domains of the rod vertical position. After substitution Eq. (6) to the linearized equation and following orthogonalizaiton with coordinate functions $\varphi_i\left(\zeta\right)$ in the range $\zeta\in[0,1]$, we deduce system of n ordinary differential equations for amplitude functions $q_i\left(\tau\right)$ (in Cauchy form):

$$\begin{bmatrix}\dot{\bar{\mathrm{x}}}_1\\ \dot{\bar{\mathrm{x}}}_2\end{bmatrix}=\begin{bmatrix}\mathbf{N} & \mathbf{E}\\ -\mathbf{I}_1^{-1}\left(\mathbf{I}_2+p_\Sigma\mathbf{I}_3\right) & -2\psi\mathbf{E}-2\psi_I\mathbf{I}_1^{-1}\mathbf{I}_2\end{bmatrix}\begin{bmatrix}\bar{\mathrm{x}}_1\\ \bar{\mathrm{x}}_2\end{bmatrix} \tag{7}$$

where $\overline{x}_1 = \overline{q}$, $\overline{x}_2 = \dot{\overline{q}}$, $\overline{q}^T = [q_1, \ldots q_i, \ldots q_n]$, **N** and **E** – accordingly null and unit quadratic matrices with dimension n, $\mathbf{I}_k = \{J_{kij} \mid i, j = \overline{1,n}\}$, $k = \overline{1,3}$ - quadratic symmetric matrix with dimension n (matrix $\mathbf{I}_1$ should be normalized, $\mathbf{I}_1 = \mathbf{E}$), J_{kij} are numbers – definite integrals of coordinate functions $\varphi_i(\zeta)$ and their derivatives:

$$J_{1ij} = \int_0^1 \varphi_j \varphi_i d\zeta, \quad J_{2ij} = \int_0^1 \varphi_j'' \varphi_i'' d\zeta, \quad J_{3ij} = -\int_0^1 \varphi_j' \varphi_i' (1-\zeta) d\zeta .$$

Forms of rod transverse oscillations determining from equation $\varphi^{IV} - \alpha^4 \varphi = 0$ and respective boundary conditions are taken as coordinate functions. Solution of the eigenmodes problem of the cantilevered rod is produced with help of Krylov's functions [4]:

$$\varphi_i(\zeta) = K_2(\alpha_i) K_3(\alpha_i \zeta) - K_1(\alpha_i) K_4(\alpha_i \zeta), \quad i = \overline{1,n} .$$

Let us limit parameter domain until third main resonance for that we take into account first six modes $n = 6$ in approximate solution (6).

Thus, we have five variable parameters for system of equations: frequency and amplitude of excitation; parameter of rod flexibility; external and internal damping.

Modeling results

Two-parameter domains where stabilization of rectilinear form of the rod's axis is possible were obtained as the result of numeric simulation. Stability analysis is performed according to Floquet - Lyapunov's theory. As a stability criterion it is taken inequality $\max|\mu_i| < 1$, $i = \overline{1,2n}$, where μ_i are multipliers.

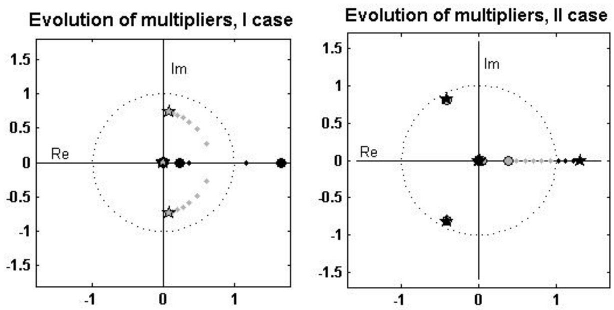
Evolution of multipliers for fixed parameters of flexibility $\gamma = 8 > \gamma_{crit}$ (supercritical rod), external and internal damping $\psi = 0, \psi_i = 0.01$ and amplitude of excitation $\beta = 0.25$ for frequency intervals $\omega \in [3,5]; [15,20]; [20,25]; [25,27]$ near unit circle on complex plane is shown at Fig. 2.

At first frequency interval (Fig. 2a), initial magnitude of one of the multipliers (solid circles at figure) sets out of the unit circle that corresponds to static unstable state (black color) when rod is bended under its own weight. With increase of excitation magnitude the multiplier crosses unit circle at the point $\{+1,0\}$, its end position (star at figure) corresponds to the case when vertical form of the rod's axis is stable (grey color). Amplitude – frequency of excitation parameter domain corresponding to similar behavior of multipliers is shown at Fig. 3a (white regions – stable state). Low boundary of stability domain (Fig. 3a – black curve) satisfies next inequality:

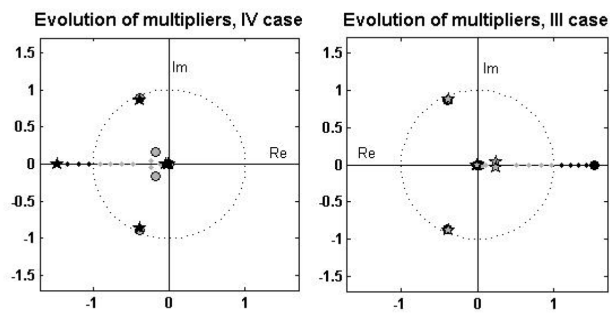
$$\beta\Omega > \sqrt{\frac{2}{|J_{311}|}(\gamma - \gamma_{Crit})}, \quad \beta_{Crit} = \frac{1}{\Omega}\sqrt{\frac{2}{|J_{311}|}(\gamma - \gamma_{Crit})} \tag{10}$$

(this boundary can be determined with use of asymptotic approximation [3] or technique of movement averaging [5]).

At further increase of excitation frequency, transition between stable-unstable values of trivial solution is (8) observed on pattern of evolution of multipliers (Fig. 2b, c, d). This cases correspond to subresonance at excitation frequency $\omega_{sub} = p_2 = 22$ and to main parametric resonance at $\omega_2 = 2p_2 = 44$ (where eigen frequency of bending vibration $p_i = \alpha_i^2 = \sqrt{J_{2ii}/J_{1ii}}$, $p_1 = 3.5$, $p_2 = 22$).



a) $\omega \in [3,5]$ b) $\omega \in [15,20]$



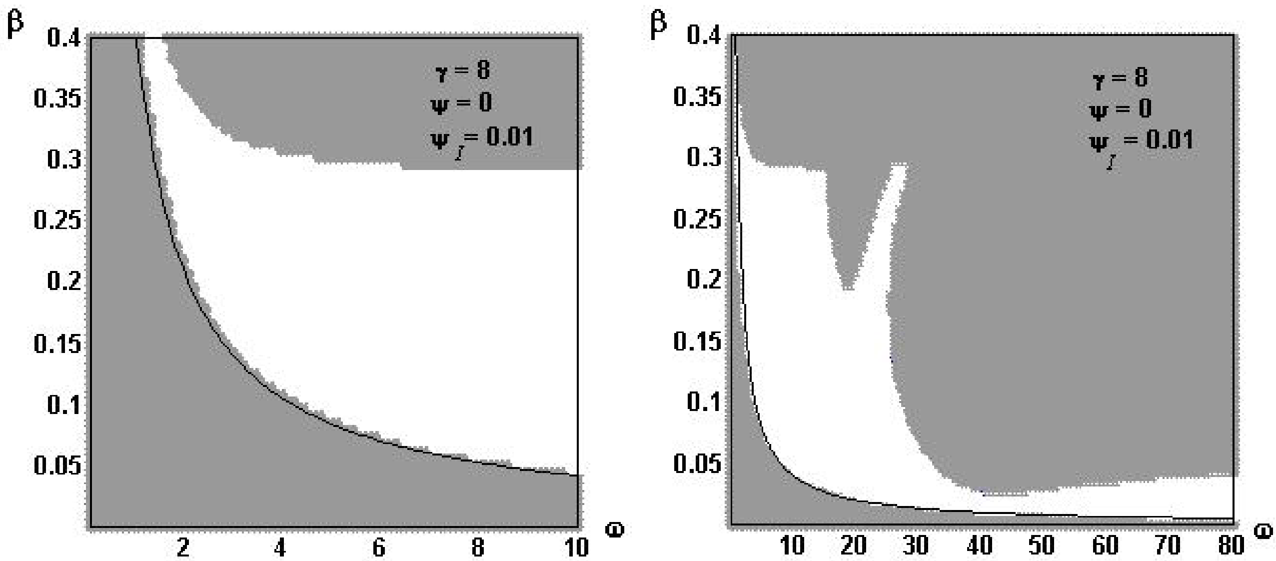
c) $\omega \in [20,25]$ d) $\omega \in [25,27]$

Fig. 2. Evolution of multipliers, $\gamma = 8 > \gamma_{crit}$, $\psi = 0, \psi_i = 0.01$, $\beta = 0.25$

Multipliers cross unit circle through points $\{\pm 1,0\}$ on boundaries. Wider instability domain for excitation parameters is shown at Fig. 3 b.

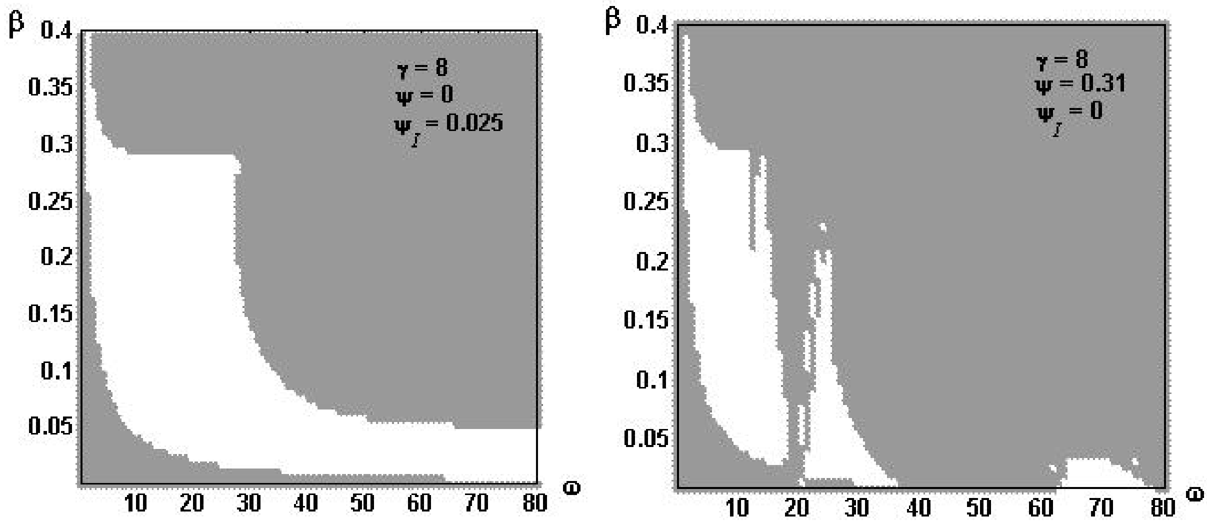
Let us consider the influence of damping on the stabilization of rectilinear form of the rod's vertical axle. Adding of linear damping increases stability regions. It is evident on comparison of instability diagrams at $\psi_I = 0.01$ (Fig. 2a) and $\psi_I = 0.025$ (Fig. 3a). Presence of external friction (Fig. 4,b - $\psi = 0.31, \psi_I = 0$) relative to internal (Fig. 3,b - $\psi = 0, \psi_I = 0.01$) confines the stable region of parameters distinctly (with equivalence of friction: $\psi \approx \psi_I \alpha_1^4 = \psi_I J_{211}$). External friction provides destabilize influence.

One can analyze influence of the flexibility parameter γ on stabilization in consideration of flexibility – excitation frequency parameters domain (Fig. 5). Unstable value of trivial solution is observed near resonance ($\omega_1 = 7, \omega_2 = 44$) and subresonance ($\omega_{sub} = 22$) frequencies. At that, multipliers at instability boundaries cross unit circle at the points $\{\pm 1, 0\}$. It is available one more tongue at excitation frequency $\omega = 25$. At this case, multipliers behave as at Fig. 5 b, namely, they cross unit circle at nonzero real and imaginary value points. Such type of bifurcation corresponds to *combination resonance*. Note, the rod in the absence of external excitation (static condition) buckles at critical value of flexibility parameter $\gamma_{crit} = 7{,}8$ (Fig. 5a), that complies with known analytical solution [1].



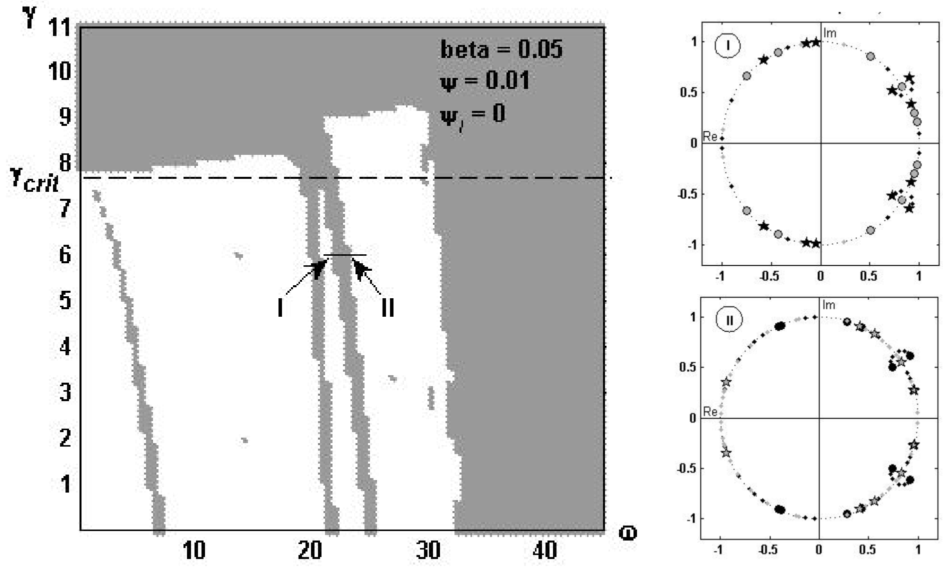
a) b)

Fig. 3. Instability domains of the supercritical rod



a) b)

Fig. 4. Instability domains of the supercritical rod



a) b)

Fig. 5: a) instability domain of the rod vertical position,
b) evolution of multipliers on boundaries of resonance-tongue interaction

Conclusion

Stabilization conditions of rectilinear form of the rod's axis were determined on the research of continual model of the flexible rod under parametric excitation. Stability domains were showed up, behavior of their boundaries was investigated and values of parametric resonances were determined.

Influence of damping on stabilization of rectilinearity of the rod's vertical axis was ascertained. Presence of external friction relative to internal limits stability domain of excitation parameters distinctly.

Acknowledgments

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