326. Dynamic synchronization of the unbalanced rotors for the excitation of longitudinal traveling waves

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Abstract. The problem about the synchronization of the unbalanced rotors for the excitation of the longitudinal traveling waves in the elastic system of a bar type is examined. The steady-state regimes of motion are investigated, the conditions of their existence and stability are obtained.

Keywords: synchronization, elastic system, longitudinal traveling waves.

1. Introduction

The study of the excitation methods of prescribed vibrations of a certain elastic system presents one of the important problems of the theory of vibration machines and devices. In this case the problem about the synchronous operation of several vibroexciters, connected with the united oscillatory system arises [1].

In the present work the problem about the synchronization of several vibration exciters of the elastic system for the excitation of the longitudinal traveling waves is investigated. The equations of the problem are nonlinear. For the solution the small parameter which enables to use methods of the theory of periodic solutions of nonlinear differential equations is introduced.

2. The formulation of the problem

The system consisting of a semi-infinite bar with the elastically connected end and n vibration exciters connected to it is analysed.

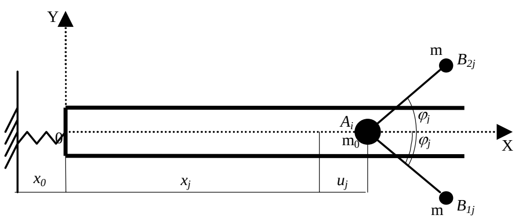

Fig. 1. Model of the system

The body of the vibration exciter is attached to the bar at the point A_j, where A_j - the center of the axis of rotation of the rotors of vibroexciters. The exciting masses of vibration exciter are located at points B_{1j} and B_{2j}, and they rotate synchronously in opposite directions so that the excitation along the bar is created [2].

It is designated $r = \overline{A_j B_{1j}} = \overline{A_j B_{2j}}$, $j = 1, \dots, n$.

According to Fig. 1 the points A_j, B_{1j}, B_{2j} have the coordinates

$$A_j(x_j + u_j, 0),\ B_{1j}(x_j + u_j + r\cos\varphi_j, -r\sin\varphi_j),$$
$$B_{2j}(x_j + u_j + r\cos\varphi_j, r\sin\varphi_j),\ x_j = const.$$

Kinetic energy of the j-th vibroexciter is equal to

$$T_j = \frac{1}{2}(m_0 + 2m)(\dot{x}_0 + \dot{u}_j)^2 + (J + mr^2)\dot{\varphi}_j^2 - $$
$$-2mr(\dot{x}_0 + \dot{u}_j)\dot{\varphi}_j \sin\varphi_j, \qquad (1)$$

where m_0 is the mass, concentrated at the point A_j, and m are the masses, concentrated at points B_{1j} and B_{2j}, J is the moment of inertia of the rotor of the j-th vibration exciter with respect to the center of the axis of rotation A_j.

The equations of motion of this system have the form

$$2(J+mr^2)\ddot{\varphi}_j - 2mr(\ddot{x}_0+\ddot{u}_j)\sin\varphi_j + H_{\varphi j}\dot{\varphi}_j = M_{\varphi j}\,, \qquad (2)$$

$$(m_0+2m)(\ddot{x}_0+\ddot{u}_j) - 2mr(\ddot{\varphi}_j\sin\varphi_j + \dot{\varphi}_j^2\cos\varphi_j) = F_{in_j}\,, \quad (3)$$

where $H_{\varphi j}\dot{\varphi}_j$ are dissipative forces, $H_{\varphi j}$ - coefficients of viscous friction for the rotation of the case of the j-th vibration exciter with respect to the axis A_j, $M_{\varphi j}$ are the moments of external forces, F_{in_j} are inertial forces.

The longitudinal vibrations of a bar are described by the equation [4]

$$EF\frac{\partial^2 u}{\partial x^2} + \xi\frac{\partial u}{\partial t} - \rho F\left(\frac{\partial^2 u}{\partial t^2} + \ddot{x}_0\right) = \sum_j \delta(x-x_j)F_{in_j}\,, \qquad (4)$$

where $u(x,t)$ is the displacement of the cross section with the abscissa x, ρ is the mass of the unit of volume, E - the modulus of elasticity of the material, F - cross-sectional area, ξ - coefficient characterizing external damping, δ - Dirac's delta function.

For the elastically fixed end of the bar the boundary condition has the following form

$$c_0x_0 = EF\frac{\partial u}{\partial x}\,, \qquad (5)$$

where c_0 is the coefficient of stiffness of the spring.

By introducing the dimensionless coordinate $\eta = \dfrac{x}{r}$, the equation (4) and the condition (5) take the form

$$\frac{EF}{r^2}\frac{\partial^2 u}{\partial \eta^2} + \xi\frac{\partial u}{\partial t} - \rho F\left(\frac{\partial^2 u}{\partial t^2} + \ddot{x}_0\right) = \sum_j \delta(\eta-\eta_j)F_{in_j}\,, \qquad (6)$$

$$c_0x_0 = \frac{EF}{r}\frac{\partial u}{\partial \eta}\,. \qquad (7)$$

Thus, equations (2), (3), (6) with the condition (7) are the differential equations of motion for this system.

3. Steady-state regimes of motion

The method of small parameter is used for investigation of the steady-state regimes of motion, determined by equations (2), (3), (6). Then the equation (2) takes the form

$$2(J+mr^2)\ddot{\varphi}_j = \varepsilon\Phi_j\,, \qquad (8)$$

where $\Phi_j = 2mr(\ddot{x}_0+\ddot{u}_j)\sin\varphi_j - H_{\varphi j}\dot{\varphi}_j + M_{\varphi j}\,,$ (9)

ε is the small parameter, at the end of the calculations assumed equal to one.

The steady-state regimes of motion are represented in the form

$$\begin{aligned} u_j &= u_{j0} + \varepsilon u_{j1} + \varepsilon^2 u_{j2} + \dots, \\ \varphi_j &= \varphi_{j0} + \varepsilon\varphi_{j1} + \varepsilon^2\varphi_{j2} + \dots, \\ F_{in_j} &= F_{in_{j0}} + \varepsilon F_{in_{j1}} + \varepsilon^2 F_{in_{j2}} + \dots, \end{aligned} \qquad (10)$$

u_j, φ_j, F_{in_j} are periodic functions of t.

Substituting (10) into the equations (8), (3), (6) and equalizing coefficients with the identical degrees of ε in the right and left sides of the equality, the differential equations for the determination of $u_{jk}, \varphi_{jk}, F_{in_{jk}}$, $k = 0,1,2,\dots$ are obtained.

Substitution of (10) into equation (8) and equating of coefficients at ε^0 and ε^1 gives the equations of the zero and first approximation

$$2(J+mr^2)\ddot{\varphi}_{j0} = 0\,, \qquad (11)$$

$$2(J+mr^2)\ddot{\varphi}_{j1} = \Phi_{j0}\,, \qquad (12)$$

where

$$\Phi_{j0} = \Phi_j\Big|_{\substack{u_j=u_{j0}\\ \varphi_j=\varphi_{j0}}} = 2mr(\ddot{x}_0+\ddot{u}_{j0})\sin\varphi_{j0} - H_{\varphi j}\dot{\varphi}_{j0} + M_{\varphi j}\,.$$

It follows that φ_{j0} can be represented in the following form

$$\varphi_{j0} = \omega t + \overline{\varphi}_{j0}\,, \qquad (13)$$

where $\overline{\varphi}_{j0} = const$.

Equation (12) is the differential equation for the determination of φ_{j1}. Periodicity condition of φ_{j1} according to the equation (12) has the form

$$\overline{\Phi}_{j0} = \overline{2mr(\ddot{x}_0+\ddot{u}_{j0})\sin(\omega t+\overline{\varphi}_{j0}) - H_{\varphi j}\omega + M_{\varphi j}} = 0\,, \quad (14)$$

the upper dash indicates averaging with respect to t.

Constants ω and $\overline{\varphi}_{j0}$, $j = 1,\dots,n$, are determined from the condition (14), but before it is necessary to find the functions u_{j0}, $j = 1,\dots,n$, from (6), (3). Substituting (10) with $\varphi_{j0} = \omega t + \overline{\varphi}_{j0}$ in (6) and (3) and equating the coefficients at ε^0, it is obtained

$$\frac{EF}{r^2}\frac{\partial^2 u}{\partial \eta^2} + \xi\frac{\partial u}{\partial t} - \rho F\left(\frac{\partial^2 u}{\partial t^2} + \ddot{x}_0\right) = \sum_j \delta(\eta-\eta_j)F_{in_{j0}}\,, \quad (15)$$

$$F_{in_{j0}} = (m_0+2m)(\ddot{x}_0+\ddot{u}_{j0}) - 2mr\omega^2\cos(\omega t+\overline{\varphi}_{j0})\,. \quad (16)$$

4. Solution of the problem describing the forced longitudinal vibrations of the bar

Further the determination of the solutions $u(\eta,t)$ of the equation (15) with the boundary condition (7) is described. At the points with the abscissa x_j (or in the dimensionless coordinates η_j) of the bar the forces F_{in_j0} (16) are applied.

Using the condition (7), the equation (15) is rewritten in the following form

$$\frac{EF}{r^2}\frac{\partial^2 u}{\partial\eta^2}+\xi\frac{\partial u}{\partial t}-\rho F\left(\frac{\partial^2 u}{\partial t^2}+\frac{EF}{c_0 r}\frac{\partial^3 u}{\partial\eta\partial t^2}\right)=$$
$$=\sum_j \delta(\eta-\eta_j)F_{in_j0}. \qquad (17)$$

The case n = 2 is examined. The solution of this problem is reduced to the integration of the differential equation of the vibrations of the bar

$$\frac{EF}{r^2}\frac{\partial^2 u}{\partial\eta^2}+\xi\frac{\partial u}{\partial t}-\rho F\left(\frac{\partial^2 u}{\partial t^2}+\frac{EF}{c_0 r}\frac{\partial^3 u}{\partial\eta\partial t^2}\right)=0,$$
$$\eta\neq\eta_1, \eta\neq\eta_2, \qquad (18)$$

valid everywhere, except at the points $\eta=\eta_1$ and $\eta=\eta_2$, at which the forces are applied

$$F_{in10}=(m_0+2m)(\ddot{x}_0+\ddot{u}_{10})-2mr\omega^2\cos(\omega t+\overline{\varphi}_{10})$$

and

$$F_{in20}=(m_0+2m)(\ddot{x}_0+\ddot{u}_{20})-2mr\omega^2\cos(\omega t+\overline{\varphi}_{20})$$

respectively.

Equation (18) must be integrated with the following conditions:
$u\to 0$ when $\eta\to+\infty$,

$$\begin{cases} u^{(1)}\Big|_{\eta=\eta_1}=u^{(2)}\Big|_{\eta=0}, \\ \dfrac{\partial u^{(1)}}{\partial\eta}\Bigg|_{\eta=\eta_1}=\dfrac{\partial u^{(2)}}{\partial\eta}\Bigg|_{\eta=0}+\dfrac{rF_{in10}}{EF}, \end{cases} \qquad (19)$$

and

$$\begin{cases} u^{(2)}\Big|_{\eta=\eta_2}=u^{(3)}\Big|_{\eta=0}, \\ \dfrac{\partial u^{(2)}}{\partial\eta}\Bigg|_{\eta=\eta_2}=\dfrac{\partial u^{(3)}}{\partial\eta}\Bigg|_{\eta=0}+\dfrac{rF_{in20}}{EF}, \end{cases} \qquad (20)$$

where $u^{(1)}=u$ when $0\le\eta\le\eta_1$, $u^{(2)}=u$ when $0\le\eta\le\eta_2$, $u^{(3)}=u$ when $0\le\eta<+\infty$.

The solution of equation (18) is sought in the form

$$u(\eta,t)=\theta(\eta)\cos\omega t+\psi(\eta)\sin\omega t. \qquad (21)$$

Designating

$$\mu=\frac{\xi r^2\omega}{EF},\ \beta^2=\frac{\rho r^2\omega^2}{E},\ k=\frac{EF\beta^2}{c_0 r}=\frac{\rho F r\omega^2}{c_0}$$

and substituting the expression (21) into the equation (18), the functions $\theta(\eta)$ and $\psi(\eta)$ are found

$$\begin{cases} \theta(\eta)=e^{-\frac{k}{2}\eta}(A_1e^{r_1\eta}+A_2e^{-r_1\eta}+A_3e^{r_2\eta}+A_4e^{-r_2\eta}), \\ \psi(\eta)=ie^{-\frac{k}{2}\eta}(-A_1e^{r_1\eta}-A_2e^{-r_1\eta}+A_3e^{r_2\eta}+A_4e^{-r_2\eta}), \end{cases} \qquad (22)$$

where $r_1=\alpha+i\alpha_1$, $r_2=\alpha-i\alpha_1$, α, α_1 are positive real numbers

$$\alpha=\frac{\sqrt{\sqrt{(k^2-4\beta^2)^2+16\mu^2}+(k^2-4\beta^2)}}{2\sqrt{2}},$$
$$\alpha_1=\frac{\sqrt{\sqrt{(k^2-4\beta^2)^2+16\mu^2}-(k^2-4\beta^2)}}{2\sqrt{2}}. \qquad (23)$$

The function $u^{(1)}(\eta,t)=\theta_1(\eta)\cos\omega t+\psi_1(\eta)\sin\omega t$ is determined for $\eta\in[0,\eta_1]$, therefore, the functions $\theta_1(\eta)$, $\psi_1(\eta)$ must satisfy the conditions

$$\theta_1(0)=\frac{k}{\beta^2}\frac{\partial\theta_1}{\partial\eta}(0),\ \psi_1(0)=\frac{k}{\beta^2}\frac{\partial\psi_1}{\partial\eta}(0), \qquad (24)$$

obtained from (7);
the function $u^{(3)}(\eta,t)=\theta_3(\eta)\cos\omega t+\psi_3(\eta)\sin\omega t$ approaches zero when $\eta\to+\infty$, i. e.

$$\theta_3(\eta)\to 0,\ \psi_3(\eta)\to 0 \text{ when } \eta\to+\infty. \qquad (25)$$

Besides, the functions $u^{(1)}(\eta,t)$, $u^{(3)}(\eta,t)$ and $u^{(2)}(\eta,t)=\theta_2(\eta)\cos\omega t+\psi_2(\eta)\sin\omega t$ must satisfy the conditions (19) and (20).

Substituting conditions (24) in (22), it is obtained

$$\begin{cases} \theta_1(\eta)=e^{-\frac{k}{2}\eta}\left(A_{11}\left[e^{r_1\eta}+\dfrac{kr_1-k^2/2-\beta^2}{kr_1+k^2/2+\beta^2}e^{-r_1\eta}\right]+\right. \\ \qquad\left.+A_{12}\left[e^{r_2\eta}+\dfrac{kr_2-k^2/2-\beta^2}{kr_2+k^2/2+\beta^2}e^{-r_2\eta}\right]\right), \\ \psi_1(\eta)=ie^{-\frac{k}{2}\eta}\left(-A_{11}\left[e^{r_1\eta}+\dfrac{kr_1-k^2/2-\beta^2}{kr_1+k^2/2+\beta^2}e^{-r_1\eta}\right]+\right. \\ \qquad\left.+A_{12}\left[e^{r_2\eta}+\dfrac{kr_2-k^2/2-\beta^2}{kr_2+k^2/2+\beta^2}e^{-r_2\eta}\right]\right). \end{cases} \qquad (26)$$

It is assumed that

$$\begin{cases} \theta_2(\eta) = e^{-\frac{k}{2}\eta}(A_{21}e^{r_1\eta} + A_{22}e^{-r_1\eta} + A_{23}e^{r_2\eta} + A_{24}e^{-r_2\eta}), \\ \psi_2(\eta) = \\ = ie^{-\frac{k}{2}\eta}(-A_{21}e^{r_1\eta} - A_{22}e^{-r_1\eta} + A_{23}e^{r_2\eta} + A_{24}e^{-r_2\eta}). \end{cases} \tag{27}$$

With the satisfaction of conditions (25) when $\eta \to +\infty$, it is obtained

$$\begin{cases} \theta_3(\eta) = e^{-\frac{k}{2}\eta}(A_{31}e^{-r_1\eta} + A_{32}e^{-r_2\eta}), \\ \psi_3(\eta) = ie^{-\frac{k}{2}\eta}(-A_{31}e^{-r_1\eta} + A_{32}e^{-r_2\eta}). \end{cases} \tag{28}$$

Using the connection conditions obtained from boundary conditions (19) and (20) it follows that

$$\begin{cases} \theta_1\big|_{\eta=\eta_1} = \theta_2\big|_{\eta=0}, \quad \psi_1\big|_{\eta=\eta_1} = \psi_2\big|_{\eta=0}, \\ \left.\dfrac{\partial\theta_1}{\partial\eta}\right|_{\eta=\eta_1} = \left.\dfrac{\partial\theta_2}{\partial\eta}\right|_{\eta=0} - a\theta_1(\eta_1) - \dfrac{ak}{\beta^2}\dfrac{\partial\theta_1}{\partial\eta}(\eta_1) - b\cos\overline{\varphi}_{10}, \\ \left.\dfrac{\partial\psi_1}{\partial\eta}\right|_{\eta=\eta_1} = \left.\dfrac{\partial\psi_2}{\partial\eta}\right|_{\eta=0} - a\psi_1(\eta_1) - \dfrac{ak}{\beta^2}\dfrac{\partial\psi_1}{\partial\eta}(\eta_1) + b\sin\overline{\varphi}_{10}, \end{cases}$$

$$\begin{cases} \theta_2\big|_{\eta=\eta_2} = \theta_3\big|_{\eta=0}, \quad \psi_2\big|_{\eta=\eta_3} = \psi_3\big|_{\eta=0}, \\ \left.\dfrac{\partial\theta_2}{\partial\eta}\right|_{\eta=\eta_2} = \left.\dfrac{\partial\theta_3}{\partial\eta}\right|_{\eta=0} - a\theta_2(\eta_2) - \dfrac{ak}{\beta^2}\dfrac{\partial\theta_2}{\partial\eta}(\eta_2) - b\cos\overline{\varphi}_{20}, \\ \left.\dfrac{\partial\psi_2}{\partial\eta}\right|_{\eta=\eta_2} = \left.\dfrac{\partial\psi_3}{\partial\eta}\right|_{\eta=0} - a\psi_2(\eta_2) - \dfrac{ak}{\beta^2}\dfrac{\partial\psi_2}{\partial\eta}(\eta_2) + b\sin\overline{\varphi}_{20}, \end{cases}$$

where $a = \dfrac{m_0 + 2m}{EF} r\omega^2 = a_1\beta^2$, $a_1 = \dfrac{m_0 + 2m}{\rho F \cdot r}$,

$b = \dfrac{2m}{EF} r^2\omega^2 = b_1\beta^2$, $b_1 = \dfrac{2m}{\rho F}$,

the integration constants $A_{11} - A_{12}$, $A_{21} - A_{24}$, $A_{31} - A_{32}$ in (26) – (28) are determined.

Functions $u^{(k)}(\eta, t)$, $k = 1,2,3$, are represented in the form

$$u^{(k)}(\eta, t) = A^{(k)}(\eta)\cos(\omega t + \gamma^{(k)}(\eta)), \tag{29}$$

where the amplitude of vibrations of the points of a bar

$$A^{(k)}(\eta) = \sqrt{\theta_k^2(\eta) + \psi_k^2(\eta)},$$

and the initial phase

$$\gamma^{(k)}(\eta) = -arctg\frac{\psi_k(\eta)}{\theta_k(\eta)}.$$

Presenting $u^{(1)}(\eta, t)$ in the form (29), it is obtained

$$\begin{aligned} u^{(1)}(\eta, t) &= A^{(11)}(\eta)\cos(\omega t + \overline{\varphi}_{20} + \gamma^{(11)}(\eta)) + \\ &+ A^{(12)}(\eta)\cos(\omega t + \overline{\varphi}_{10} + \gamma^{(12)}(\eta)) + \\ &+ A^{(13)}(\eta)\cos(\omega t + \overline{\varphi}_{10} + \gamma^{(13)}(\eta)), \quad 0 \le \eta \le \eta_1, \end{aligned} \tag{30}$$

where $A^{(11)}(\eta) = \dfrac{b\sqrt{\alpha^2 + \alpha_1^2}}{\sqrt{2}\Delta} e^{\frac{k}{2}(\eta_1 + \eta_2 - \eta)} e^{-\alpha(\eta_1 + \eta_2)} M_1(\eta)$,

$$A^{(12)}(\eta) = \frac{b\sqrt{\alpha^2 + \alpha_1^2}}{\sqrt{2}\Delta} e^{\frac{k}{2}(\eta_1 - \eta)} e^{-\alpha\eta_1} M_1(\eta),$$

$$A^{(13)}(\eta) = \frac{ab}{2\beta^2\Delta} e^{\frac{k}{2}(\eta_1 - \eta)} e^{-\alpha(\eta_1 + \eta_2)} M_1(\eta) M_2(\eta_2),$$

$$\gamma^{(11)}(\eta) = \alpha_1(\eta_1 + \eta_2) - arctg\frac{\alpha_1}{\alpha} + \Lambda + arctg\Lambda_1(\eta),$$

$$\gamma^{(12)}(\eta) = \alpha_1\eta_1 - arctg\frac{\alpha_1}{\alpha} + \Lambda + arctg\Lambda_1(\eta),$$

$$\gamma^{(13)}(\eta) = \alpha_1(\eta_1 + \eta_2) - arctg\Lambda_2(\eta_2) + \Lambda + arctg\Lambda_1(\eta);$$

$$M_1(\eta) = \sqrt{\frac{\lambda_1 ch2\alpha\eta + \lambda_2 sh2\alpha\eta + \lambda_3\cos 2\alpha_1\eta + \lambda_4\sin 2\alpha_1\eta}{\lambda_1 + \lambda_2}},$$

$$M_2(\eta) = \sqrt{\mu_1 ch2\alpha\eta - \mu_2 sh2\alpha\eta + \mu_3\cos 2\alpha_1\eta - \mu_4\sin 2\alpha_1\eta},$$

$$\lambda_1 = k^2(\alpha^2 + \alpha_1^2) + \left(k^2/2 + \beta^2\right)^2, \ \lambda_2 = 2k\alpha\left(k^2/2 + \beta^2\right),$$

$$\lambda_3 = k^2(\alpha^2 + \alpha_1^2) - \left(k^2/2 + \beta^2\right)^2, \ \lambda_4 = 2k\alpha_1\left(k^2/2 + \beta^2\right),$$

$$\mu_1 = k^2(\alpha^2 + \alpha_1^2) + \left(k^2/2 - \beta^2\right)^2, \ \mu_2 = 2k\alpha\left(k^2/2 - \beta^2\right),$$

$$\mu_3 = k^2(\alpha^2 + \alpha_1^2) - \left(k^2/2 - \beta^2\right)^2, \ \mu_4 = 2k\alpha_1\left(k^2/2 - \beta^2\right),$$

$$\Lambda_1(\eta) = \frac{\nu_1(\eta)}{\nu_2(\eta)},$$

$$\nu_1(\eta) = (-(\lambda_1 + \lambda_2)e^{\alpha\eta} + \lambda_3 e^{-\alpha\eta})tg\alpha_1\eta - \lambda_4 e^{-\alpha\eta},$$

$$\nu_2(\eta) = (\lambda_1 + \lambda_2)e^{\alpha\eta} + \lambda_3 e^{-\alpha\eta} + \lambda_4 e^{-\alpha\eta} tg\alpha_1\eta,$$

$$\Lambda_2(\eta) = \frac{k\alpha_1 + \left(k\alpha \cdot th\alpha\eta - \left(k^2/2 - \beta^2\right)\right)tg\alpha_1\eta}{k\alpha - \left(k\alpha_1 \cdot tg\alpha_1\eta + \left(k^2/2 - \beta^2\right)\right)th\alpha\eta},$$

$$\Delta = \sqrt{(B_1B_2 - C_1C_2 - D_1)^2 + (B_1C_2 + C_1B_2 - D_2)^2},$$

$$\Lambda = arctg\frac{B_1C_2 + C_1B_2 - D_2}{B_1B_2 - C_1C_2 - D_1},$$

$$B_1 = \alpha + \frac{a_1}{2}\mu_5 - \frac{a_1 e^{-2\alpha\eta_1}}{2(\lambda_1 + \lambda_2)}[\lambda_5\cos 2\alpha_1\eta_1 + \lambda_6\sin 2\alpha_1\eta_1],$$

$$C_1 = \alpha_1 + \frac{a_1}{2}k\alpha_1 - \frac{a_1 e^{-2\alpha\eta_1}}{2(\lambda_1 + \lambda_2)}[\lambda_6\cos 2\alpha_1\eta_1 - \lambda_6\sin 2\alpha_1\eta_1],$$

$$B_2 = \alpha + \frac{a_1}{2}\mu_5 + \frac{a_1 e^{-2\alpha\eta_2}}{2}[\mu_6\cos 2\alpha_1\eta_2 + k\alpha_1\sin 2\alpha_1\eta_2],$$

$$C_2 = \alpha_1 + \frac{a_1}{2}k\alpha_1 + \frac{a_1 e^{-2\alpha\eta_2}}{2}[k\alpha_1\cos 2\alpha_1\eta_2 - \mu_6\sin 2\alpha_1\eta_2],$$

$$D_1 = \frac{a_1 e^{-2\alpha\eta_2}}{2}[\mu_7 \cos 2\alpha_1\eta_2 + \mu_8 \sin 2\alpha_1\eta_2] +$$
$$+\frac{a_1 e^{-2\alpha(\eta_1+\eta_2)}}{2(\lambda_1+\lambda_2)}[\lambda_7 \cos 2\alpha_1(\eta_1+\eta_2) + \lambda_8 \sin 2\alpha_1(\eta_1+\eta_2)],$$
$$D_2 = \frac{a_1 e^{-2\alpha\eta_2}}{2}[\mu_8 \cos 2\alpha_1\eta_2 - \mu_7 \sin 2\alpha_1\eta_2] +$$
$$+\frac{a_1 e^{-2\alpha(\eta_1+\eta_2)}}{2(\lambda_1+\lambda_2)}[\lambda_8 \cos 2\alpha_1(\eta_1+\eta_2) - \lambda_7 \sin 2\alpha_1(\eta_1+\eta_2)],$$
$$\mu_5 = k\alpha - k^2/2 + \beta^2, \ \lambda_5 = \lambda_3(k\alpha + k^2/2 - \beta^2) - \lambda_4 \cdot k\alpha_1,$$
$$\mu_6 = k\alpha + k^2/2 - \beta^2, \ \lambda_6 = \lambda_3 \cdot k\alpha_1 + \lambda_4(k\alpha + k^2/2 - \beta^2),$$
$$\mu_7 = k(\alpha^2 - \alpha_1^2) + \alpha\left(k^2/2 - \beta^2\right), \ \lambda_7 = \lambda_3\mu_7 - \lambda_4\mu_8,$$
$$\mu_8 = 2k\alpha\alpha_1 + \alpha_1\left(k^2/2 - \beta^2\right), \ \lambda_8 = \lambda_3\mu_8 + \lambda_4\mu_7.$$

Now $u^{(2)}(\eta,t)$ is represented in the form (29)

$$u^{(2)}(\eta,t) = A^{(21)}(\eta)\cos(\omega t + \overline{\varphi}_{20} + \gamma^{(21)}(\eta)) +$$
$$+ A^{(22)}(\eta)\cos(\omega t + \overline{\varphi}_{10} + \gamma^{(22)}(\eta)) +$$
$$+ A^{(23)}(\eta)\cos(\omega t + \overline{\varphi}_{10} + \gamma^{(23)}(\eta)) +$$
$$+ A^{(24)}(\eta)\cos(\omega t + \overline{\varphi}_{20} + \gamma^{(24)}(\eta)),$$
$$\eta_1 \le \eta + \eta_1 \le \eta_1 + \eta_2, \tag{31}$$

where $A^{(21)}(\eta) = \frac{b\sqrt{\alpha^2+\alpha_1^2}}{\sqrt{2}\Delta} e^{\frac{k}{2}(\eta_2-\eta)} e^{-\alpha(\eta_1+\eta_2)} M_1(\eta+\eta_1)$,

$$A^{(22)}(\eta) = \frac{b\sqrt{\alpha^2+\alpha_1^2}}{\sqrt{2}\Delta} e^{-\frac{k}{2}\eta} e^{-\alpha(\eta_1+\eta)} M_1(\eta_1),$$
$$A^{(23)}(\eta) = \frac{ab}{2\beta^2\Delta} e^{-\frac{k}{2}\eta} e^{-\alpha(\eta_1+\eta_2)} M_1(\eta_1) M_2(\eta_2-\eta),$$
$$A^{(24)}(\eta) = \frac{ab}{2\beta^2\Delta} e^{\frac{k}{2}(\eta_2-\eta)} e^{-\alpha(\eta_1+\eta_2)} M_3(\eta_1)\sqrt{ch 2\alpha\eta - \cos 2\alpha_1\eta},$$
$$\gamma^{(21)}(\eta) = \alpha_1(\eta_1+\eta_2) - arctg\frac{\alpha_1}{\alpha} + \Lambda + arctg\Lambda_1(\eta+\eta_1),$$
$$\gamma^{(22)}(\eta) = \alpha_1(\eta_1+\eta) - arctg\frac{\alpha_1}{\alpha} + \Lambda + arctg\Lambda_1(\eta_1),$$
$$\gamma^{(23)}(\eta) = \alpha_1(\eta_1+\eta_2) - arctg\Lambda_2(\eta_2-\eta) + \Lambda + arctg\Lambda_1(\eta_1),$$
$$\gamma^{(24)}(\eta) = \alpha_1(\eta_1+\eta_2) - arctg\frac{tg\alpha_1\eta}{th\alpha\eta} + \Lambda + arctg\Lambda_3(\eta_1);$$
$$M_3(\eta) = \sqrt{\frac{\chi_1 ch 2\alpha\eta + \chi_2 sh 2\alpha\eta - \chi_3\cos 2\alpha_1\eta + \chi_4 \sin 2\alpha_1\eta}{\lambda_1+\lambda_2}},$$
$$\chi_1 = \lambda_1\mu_1 - \lambda_2\mu_2, \quad \chi_2 = \lambda_2\mu_1 - \lambda_1\mu_2, \quad \chi_3 = \lambda_3\mu_3 + \lambda_4\mu_4,$$
$$\chi_4 = \lambda_3\mu_4 - \lambda_4\mu_3,$$
$$\Lambda_3(\eta) = \frac{k\alpha\cdot\nu_3(\eta) - k\alpha_1\cdot\nu_4(\eta) + \left(k^2/2-\beta^2\right)\cdot\nu_1(\eta)}{k\alpha\cdot\nu_4(\eta) + k\alpha_1\cdot\nu_4(\eta) + \left(k^2/2-\beta^2\right)\cdot\nu_2(\eta)},$$
$$\nu_3(\eta) = ((\lambda_1+\lambda_2)e^{\alpha\eta} + \lambda_3 e^{-\alpha\eta})tg\alpha_1\eta - \lambda_4 e^{-\alpha\eta},$$
$$\nu_4(\eta) = -(\lambda_1+\lambda_2)e^{\alpha\eta} + \lambda_3 e^{-\alpha\eta} + \lambda_4 e^{-\alpha\eta} tg\alpha_1\eta.$$

Representing the function $u^{(3)}(\eta,t)$ in the form (29), it is obtained

$$u^{(3)}(\eta,t) = A^{(31)}(\eta)\cos(\omega t + \overline{\varphi}_{20} + \gamma^{(31)}(\eta)) +$$
$$+ A^{(32)}(\eta)\cos(\omega t + \overline{\varphi}_{10} + \gamma^{(32)}(\eta)) +$$
$$+ A^{(33)}(\eta)\cos(\omega t + \overline{\varphi}_{20} + \gamma^{(33)}(\eta)),$$
$$\eta_1+\eta_2 \le \eta+\eta_1+\eta_2 < +\infty, \tag{32}$$

where $A^{(31)}(\eta) = \frac{b\sqrt{\alpha^2+\alpha_1^2}}{\sqrt{2}\Delta} e^{-\frac{k}{2}\eta} e^{-\alpha(\eta_1+\eta_2+\eta)} M_1(\eta_1+\eta_2)$,

$$A^{(32)}(\eta) = \frac{b\sqrt{\alpha^2+\alpha_1^2}}{\sqrt{2}\Delta}\left(1+\frac{ak}{\beta^2}\right) e^{-\frac{k}{2}(\eta+\eta_2)} e^{-\alpha(\eta_1+\eta_2+\eta)} M_1(\eta_1),$$
$$A^{(33)}(\eta) = \frac{ab}{2\beta^2\Delta} e^{-\frac{k}{2}\eta} e^{-\alpha(\eta_1+\eta_2+\eta)} M_3(\eta_1)\sqrt{ch 2\alpha\eta_2 - \cos 2\alpha_1\eta_2},$$
$$\gamma^{(31)}(\eta) = \alpha_1(\eta_1+\eta_2+\eta) - arctg\frac{\alpha_1}{\alpha} + \Lambda + arctg\Lambda_1(\eta_1+\eta_2),$$
$$\gamma^{(32)}(\eta) = \alpha_1(\eta_1+\eta_2+\eta) - arctg\frac{\alpha_1}{\alpha} + \Lambda + arctg\Lambda_1(\eta_1),$$
$$\gamma^{(33)}(\eta) = \alpha_1(\eta_1+\eta_2+\eta) - arctg\frac{tg\alpha_1\eta_2}{th\alpha\eta_2} + \Lambda + arctg\Lambda_3(\eta_1).$$

Differentiating the function $u^{(1)}(\eta,t)$ with respect to η, the value of the function $x_0(t) = \frac{k}{\beta^2}\frac{\partial u^{(1)}}{\partial\eta}(\eta_1,t)$ at $\eta=\eta_1$ is found, then adding $x_0(t)$ and $u_{10}(t) = u^{(1)}(\eta_1,t) = u^{(2)}(0,t)$, it is obtained

$$u_{10}(t) + x_0(t) = A^{(1)}\cos(\omega t + \overline{\varphi}_{20} + \gamma^{(1)}) +$$
$$+ A^{(2)}\cos(\omega t + \overline{\varphi}_{10} + \gamma^{(2)}) + A^{(3)}\cos(\omega t + \overline{\varphi}_{10} + \gamma^{(3)}), \tag{33}$$

where $A^{(1)} = \frac{b_1\sqrt{\alpha^2+\alpha_1^2}}{\sqrt{2}\Delta} e^{\frac{k}{2}\eta_2} e^{-\alpha(\eta_1+\eta_2)} M_3(\eta_1)$,

$$A^{(2)} = \frac{b_1\sqrt{\alpha^2+\alpha_1^2}}{\sqrt{2}\Delta} e^{-\alpha\eta_1} M_3(\eta_1),$$
$$A^{(3)} = \frac{a_1 b_1}{2\Delta} e^{-\alpha(\eta_1+\eta_2)} M_3(\eta_1) M_2(\eta_2),$$
$$\gamma^{(1)} = \alpha_1(\eta_1+\eta_2) - arctg\frac{\alpha_1}{\alpha} + \Lambda + arctg\Lambda_3(\eta_1),$$
$$\gamma^{(2)} = \alpha_1\eta_1 - arctg\frac{\alpha_1}{\alpha} + \Lambda + arctg\Lambda_3(\eta_1),$$
$$\gamma^{(3)} = \alpha_1(\eta_1+\eta_2) - arctg\Lambda_2(\eta_2) + \Lambda + arctg\Lambda_3(\eta_1).$$

In the same way, differentiating the function $u^{(2)}(\eta,t)$ with respect to η, the value of the function

$x_0(t)=\frac{k}{\beta^2}\frac{\partial u^{(2)}}{\partial\eta}(\eta_2,t)$ at $\eta=\eta_2$ is found, then by adding $x_0(t)$ and $u_{20}(t)=u^{(2)}(\eta_2,t)=u^{(3)}(0,t)$, it is obtained

$$u_{20}(t)+x_0(t)=A^{(4)}\cos(\omega t+\bar{\varphi}_{20}+\gamma^{(4)})-$$
$$-A^{(5)}\cos(\omega t+\bar{\varphi}_{10}+\gamma^{(5)})+A^{(3)}\cos(\omega t+\bar{\varphi}_{20}+\gamma^{(3)}), \quad (34)$$

where $A^{(4)}=\frac{b_1\sqrt{\alpha^2+\alpha_1^2}}{\sqrt{2}\Delta}e^{-\alpha(\eta_1+\eta_2)}M_3(\eta_1+\eta_2)$,

$$A^{(5)}=\frac{b_1\sqrt{\alpha^2+\alpha_1^2}}{\sqrt{2}\Delta}e^{-\frac{k}{2}\eta_2}e^{-\alpha(\eta_1+\eta_2)}\sqrt{\mu_1+\mu_2}\,M_1(\eta_1),$$

$$\gamma^{(4)}=\alpha_1(\eta_1+\eta_2)-arctg\frac{\alpha_1}{\alpha}+\Lambda+arctg\Lambda_3(\eta_1+\eta_2),$$

$$\gamma^{(5)}=\alpha_1(\eta_1+\eta_2)-arctg\frac{\alpha_1}{\alpha}+\Lambda+arctg\Lambda_1(\eta_1)-arctg\frac{k\alpha_1}{\mu_6}.$$

5. The existence and stability of the solutions

From the periodicity conditions

$$\bar{\Phi}_{j0}=\overline{2mr(\ddot{x}_0+\ddot{u}_{j0})\sin(\omega t+\bar{\varphi}_{j0})-H_{\varphi j}\omega+M_{\varphi j}}=0,\ j=1,2,$$

where the functions $u_{10}+x_0$ (33) and $u_{20}+x_0$ (34) are now already known, the constants ω and $\bar{\varphi}_{20}-\bar{\varphi}_{10}$ are determined.

Designating $h_j=\frac{H_{\varphi j}}{mr}$, $M_j^*=\frac{M_{\varphi j}}{mr}$, these conditions can be written in the form

$$\bar{\Phi}_{j0}=\overline{-2\omega^2(x_0+u_{j0})\sin(\omega t+\bar{\varphi}_{j0})-h_j\omega+M_j^*}=0. \quad (35)$$

Equations (35) by taking into account (33), (34) lead to the following expressions

$$\begin{cases}\bar{\Phi}_{10}=\omega^2\begin{pmatrix}A^{(1)}\sin(\bar{\varphi}_{20}-\bar{\varphi}_{10}+\gamma^{(1)})+\\+A^{(2)}\sin\gamma^{(2)}+A^{(3)}\sin\gamma^{(3)}\end{pmatrix}-h_1\omega+M_1^*=0,\\ \bar{\Phi}_{20}=\omega^2\begin{pmatrix}A^{(4)}\sin\gamma^{(4)}+\\+A^{(5)}\sin(\bar{\varphi}_{20}-\bar{\varphi}_{10}-\gamma^{(5)})+\\+A^{(3)}\sin\gamma^{(3)}\end{pmatrix}-h_2\omega+M_2^*=0.\end{cases} \quad (36)$$

From (36) the expression for the determination of $\bar{\varphi}_{20}-\bar{\varphi}_{10}$ is obtained:

$$\sin(\bar{\varphi}_{20}-\bar{\varphi}_{10})=\frac{F_1\sin\gamma^{(5)}+F_2\sin\gamma^{(1)}}{\sin(\gamma^{(1)}+\gamma^{(5)})}, \quad (37)$$

where $F_1=\frac{(h_1\omega-M_1^*)/\omega^2-A^{(2)}\sin\gamma^{(2)}-A^{(3)}\sin\gamma^{(3)}}{A^{(1)}}$,

$$F_2=\frac{(h_2\omega-M_2^*)/\omega^2-A^{(4)}\sin\gamma^{(4)}-A^{(3)}\sin\gamma^{(3)}}{A^{(5)}}.$$

According to (37), the condition for existence of the solutions $\bar{\varphi}_{20}-\bar{\varphi}_{10}$ is expressed by the inequality

$$\left|\frac{F_1\sin\gamma^{(5)}+F_2\sin\gamma^{(1)}}{\sin(\gamma^{(1)}+\gamma^{(5)})}\right|<1. \quad (38)$$

Values $\bar{\varphi}_{20}-\bar{\varphi}_{10}$ depend on the frequencies of excitation ω, which are determined from the equation

$$F_1^2+F_2^2-2F_1F_2\cos(\gamma^{(1)}+\gamma^{(5)})=\sin^2(\gamma^{(1)}+\gamma^{(5)}). \quad (39)$$

It follows from (39) that $|F_1|<1$, $|F_2|<1$.

Only those values of $\bar{\varphi}_{20}-\bar{\varphi}_{10}$ are stable, which satisfy the inequality

$$A^{(5)}\cos(\bar{\varphi}_{20}-\bar{\varphi}_{10}-\gamma^{(5)})-A^{(1)}\cos(\bar{\varphi}_{20}-\bar{\varphi}_{10}+\gamma^{(1)})<0. \quad (40)$$

6. Vibrations of a bar in the absence of damping

If the viscous damping is disregarded, that is $\xi=0$, then the expressions (23) for α, α_1 are represented in the form

a) when $k^2<4\beta^2$: $\alpha=0$, $\alpha_1=\frac{\sqrt{4\beta^2-k^2}}{2}$, (41)

b) when $k^2>4\beta^2$: $\alpha=\frac{\sqrt{k^2-4\beta^2}}{2}$, $\alpha_1=0$, (42)

since $\mu=0$, α, α_1 - real positive numbers.

In the case *a*) the functions $u_{10}+x_0$ (33) and $u_{20}+x_0$ (34) are transformed to the form

$$u_{10}+x_0=L_1e^{\frac{k}{2}\eta_2}\cos(\omega t+\bar{\varphi}_{20}+\beta_1)+$$
$$+L_1\cos(\omega t+\bar{\varphi}_{10}+\beta_2)+L_2\cos(\omega t+\bar{\varphi}_{10}+\beta_1), \quad (43)$$
$$u_{20}+x_0=L_3\cos(\omega t+\bar{\varphi}_{20}+\beta_1)-$$
$$-L_4e^{-\frac{k}{2}\eta_2}\sin(\omega t+\bar{\varphi}_{10}+\beta_3)+L_2\cos(\omega t+\bar{\varphi}_{20}+\beta_1), \quad (44)$$

where $L_1=\frac{b_1\alpha_1\sqrt{\mu_1}}{\Delta^{(1)}}\cos(\alpha_1\eta_1+\gamma_1+\gamma_2)$,

$$L_2=\frac{a_1b_1\mu_1}{\Delta^{(1)}}\cos(\alpha_1\eta_1+\gamma_1+\gamma_2)\cos(\alpha_1\eta_2+\gamma_2),$$

$$L_3=\frac{b_1\alpha_1\sqrt{\mu_1}}{\Delta^{(1)}}\cos(\alpha_1(\eta_1+\eta_2)+\gamma_1+\gamma_2),$$

$$L_4=\frac{b_1\alpha_1\sqrt{\mu_1}}{\Delta^{(1)}}\sin(\alpha_1\eta_1+\gamma_1),$$

$\beta_1 = \alpha_1(\eta_1 + \eta_2) + \Lambda^{(1)} + \gamma_1$,
$\beta_2 = \beta_1 - \alpha_1\eta_2$,
$\beta_3 = \beta_1 + \gamma_2$,
$\gamma_1 = arctg \dfrac{k\alpha_1}{k^2/2 + \beta^2}$,
$\gamma_2 = arctg \dfrac{k^2/2 - \beta^2}{k\alpha_1}$,
$\Delta^{(1)} = \sqrt{\Delta_1^2 + \Delta_2^2}$,
$\Lambda^{(1)} = arctg \dfrac{\Delta_1}{\Delta_2}$,

$$\Delta_1 = -a_1\alpha_1\left(\frac{k^2}{2} - \beta^2\right)\left(1 + \frac{a_1 k}{2}\right) + \frac{a_1^2\mu_1}{4}\sin 2\alpha_1\eta_2 - \frac{a_1}{2\lambda_1}\times$$
$$\times\left[\left(\alpha_1\lambda_5 - \frac{a_1\mu_1\lambda_4}{2}\right)\cos 2\alpha_1\eta_1 + \left(\alpha_1\lambda_6 + \frac{a_1\mu_1\lambda_3}{2}\right)\sin 2\alpha_1\eta_1 +\right.$$
$$\left.+\left(\alpha_1\lambda_5 + \frac{a_1\chi_4}{2}\right)\cos 2\alpha_1(\eta_1+\eta_2) + \left(\alpha_1\lambda_6 + \frac{a_1\chi_3}{2}\right)\sin 2\alpha_1(\eta_1+\eta_2)\right],$$
$$\Delta_2 = \frac{a_1^2}{4}\left(\frac{k^2}{2} - \beta^2\right)^2 - \alpha_1^2\left(1 + \frac{a_1 k}{2}\right)^2 - \frac{a_1^2\mu_1}{4}\cos 2\alpha_1\eta_2 + \frac{a_1}{2\lambda_1}\times$$
$$\times\left[\left(\alpha_1\lambda_6 + \frac{a_1\mu_1\lambda_3}{2}\right)\cos 2\alpha_1\eta_1 - \left(\alpha_1\lambda_5 - \frac{a_1\mu_1\lambda_4}{2}\right)\sin 2\alpha_1\eta_1 +\right.$$
$$\left.+\left(\alpha_1\lambda_6 + \frac{a_1\chi_3}{2}\right)\cos 2\alpha_1(\eta_1+\eta_2) - \left(\alpha_1\lambda_5 + \frac{a_1\chi_4}{2}\right)\sin 2\alpha_1(\eta_1+\eta_2)\right],$$

$\lambda_1 = k^2\alpha_1^2 + \left(k^2/2 + \beta^2\right)^2$, $\lambda_3 = k^2\alpha_1^2 - \left(k^2/2 + \beta^2\right)^2$,
$\lambda_4 = 2k\alpha_1\left(k^2/2 + \beta^2\right)$, $\lambda_5 = \lambda_3(k^2/2 - \beta^2) - \lambda_4 \cdot k\alpha_1$,
$\mu_1 = k^2\alpha_1^2 + \left(k^2/2 - \beta^2\right)^2$, $\mu_3 = k^2\alpha_1^2 - \left(k^2/2 - \beta^2\right)^2$,
$\mu_4 = 2k\alpha_1\left(k^2/2 - \beta^2\right)$, $\lambda_6 = \lambda_3 \cdot k\alpha_1 + \lambda_4(k^2/2 - \beta^2)$,
$\chi_3 = \lambda_3\mu_3 + \lambda_4\mu_4$, $\chi_4 = \lambda_3\mu_4 - \lambda_4\mu_3$.

Taking into account that $k^2 < 4\beta^2$, it is obtained that

$$k \in \left(0, \frac{4c_0 r}{EF}\right), \quad \beta^2 \in \left(0, \frac{4c_0^2 r^2}{(EF)^2}\right), \quad \alpha_1 \in \left(0, \frac{c_0 r}{EF}\right),$$
$$\omega^2 \in \left(0, \frac{4c_0^2}{EF \cdot \rho F}\right). \tag{45}$$

When $k^2/2 - \beta^2 = 0$ α_1 takes the maximum value $\dfrac{c_0 r}{EF}$,

$$k = \frac{2c_0 r}{EF}, \; \beta^2 = \frac{2c_0^2 r^2}{(EF)^2}, \; \omega^2 = \frac{2c_0^2}{EF \cdot \rho F}; \tag{46}$$

when $k^2/2 - \beta^2 < 0$: $\alpha_1 \in \left(0, \dfrac{c_0 r}{EF}\right)$, $k \in \left(0, \dfrac{2c_0 r}{EF}\right)$,

$$\beta^2 \in \left(0, \frac{2c_0^2 r^2}{(EF)^2}\right), \quad \omega^2 \in \left(0, \frac{2c_0^2}{EF \cdot \rho F}\right); \tag{47}$$

when $k^2/2 - \beta^2 > 0$: $\alpha_1 \in \left(0, \dfrac{c_0 r}{EF}\right)$, $k \in \left(\dfrac{2c_0 r}{EF}, \dfrac{4c_0 r}{EF}\right)$,

$$\beta^2 \in \left(\frac{2c_0^2 r^2}{(EF)^2}, \frac{4c_0^2 r^2}{(EF)^2}\right), \; \omega^2 \in \left(\frac{2c_0^2}{EF \cdot \rho F}, \frac{4c_0^2}{EF \cdot \rho F}\right). \tag{48}$$

Periodicity conditions lead to the expressions

$$\begin{cases} \overline{\Phi}_{10} = \omega^2\begin{pmatrix} L_1 e^{\frac{k}{2}\eta_2}\sin(\overline{\varphi}_{20} - \overline{\varphi}_{10} + \beta_1) + \\ + L_1\sin(\beta_1 - \alpha_1\eta_2) + L_2\sin\beta_1 \end{pmatrix} - h_1\omega + M_1^* = 0, \\ \overline{\Phi}_{20} = \omega^2\begin{pmatrix} L_3\sin\beta_1 + \\ + L_4 e^{-\frac{k}{2}\eta_2}\cos(\overline{\varphi}_{20} - \overline{\varphi}_{10} - \beta_1 - \gamma_2) + \\ + L_2\sin\beta_1 \end{pmatrix} - \\ - h_2\omega + M_2^* = 0. \end{cases} \tag{49}$$

From (49) it is obtained

$$\sin(\overline{\varphi}_{20} - \overline{\varphi}_{10}) = \frac{F_{11}\cos(\beta_1 + \gamma_2) - F_{12}\sin\beta_1}{\cos(2\beta_1 + \gamma_2)}, \tag{50}$$
$$F_{11}^2 + F_{12}^2 - 2F_{11}F_{12}\sin(2\beta_1 + \gamma_2) = \cos^2(2\beta_1 + \gamma_2), \tag{51}$$

where

$$F_{11} = \frac{\left((h_1\omega - M_1^*)/\omega^2 - L_1\sin(\beta_1 - \alpha_1\eta_2) - L_2\sin\beta_1\right)e^{-\frac{k}{2}\eta_2}}{L_1},$$
$$F_{12} = \frac{\left((h_2\omega - M_2^*)/\omega^2 - (L_3 + L_2)\sin\beta_1\right)e^{\frac{k}{2}\eta_2}}{L_4},$$

moreover, one is to take into account that according to (45) $\omega^2 \in \left(0, \dfrac{4c_0^2}{EF \cdot \rho F}\right)$. In (51) $|F_{11}| < 1$, $|F_{12}| < 1$.

According to (50) the condition of existence is expressed by the inequality

$$\left|\frac{F_{11}\cos(\beta_1 + \gamma_2) - F_{12}\sin\beta_1}{\cos(2\beta_1 + \gamma_2)}\right| < 1. \tag{52}$$

The stability condition of those regimes of motion which satisfy the condition of existence is the inequality

$$L_1 e^{\frac{k}{2}\eta_2}\cos(\overline{\varphi}_{20}-\overline{\varphi}_{10}+\beta_1)+ \\ +L_4 e^{-\frac{k}{2}\eta_2}\sin(\overline{\varphi}_{20}-\overline{\varphi}_{10}-\beta_1-\gamma_2)>0. \qquad (53)$$

In the case *b*) the functions $u_{10}+x_0$ (33) and $u_{20}+x_0$ (34) take the form

$$u_{10}+x_0=L_1^*\cos(\omega t+\overline{\varphi}_{20})+(L_2^*+L_3^*)\cos(\omega t+\overline{\varphi}_{10}), \\ u_{20}+x_0=(L_3^*+L_4^*)\cos(\omega t+\overline{\varphi}_{20})-L_5^*\cos(\omega t+\overline{\varphi}_{10}), \qquad (54)$$

where $L_1^*=\frac{b_1\alpha}{2\Delta^{(2)}}e^{\left(\frac{k}{2}-\alpha\right)\eta_2}s_1(s_2-s_3e^{-2\alpha\eta_1})$,

$$L_2^*=\frac{b_1\alpha}{2\Delta^{(2)}}s_1(s_2-s_3e^{-2\alpha\eta_1}),$$

$$L_3^*=\frac{a_1b_1}{4\Delta^{(2)}}s_1^2(s_2-s_3e^{-2\alpha\eta_1})(s_2+e^{-2\alpha\eta_2}),$$

$$L_4^*=\frac{b_1\alpha}{2\Delta^{(2)}}s_1(s_2-s_3e^{-2\alpha(\eta_1+\eta_2)}),$$

$$L_5^*=\frac{b_1\alpha}{2\Delta^{(2)}}e^{-\left(\frac{k}{2}+\alpha\right)\eta_2}s_1(1+s_3e^{-2\alpha\eta_1}), \qquad s_1=k\alpha+k^2/2-\beta^2,$$

$$s_2=\frac{k\alpha-k^2/2+\beta^2}{k\alpha+k^2/2-\beta^2}, \qquad s_3=\frac{k\alpha-k^2/2-\beta^2}{k\alpha+k^2/2+\beta^2},$$

$$\Delta^{(2)}=\left(\alpha+\frac{a_1}{2}s_1(s_2-s_3e^{-2\alpha\eta_1})\right)\left(\alpha+\frac{a_1}{2}s_1(s_2+e^{-2\alpha\eta_2})\right)- \\ -\frac{a_1\alpha}{2}s_1(e^{-2\alpha\eta_2}+s_3e^{-2\alpha(\eta_1+\eta_2)}).$$

By taking into account that $k^2>4\beta^2$, it is obtained that

$$k\in\left(\frac{4c_0r}{EF},+\infty\right), \qquad \beta^2\in\left(\frac{4c_0^2r^2}{(EF)^2},+\infty\right), \qquad \alpha_1\in(0,+\infty), \\ \omega^2\in\left(\frac{4c_0^2}{EF\cdot\rho F},+\infty\right). \qquad (55)$$

The averaged values $\overline{\Phi}_{i0}$, $i=1,2$, are represented in the form

$$\begin{cases}\overline{\Phi}_{10}=\omega^2L_1^*\sin(\overline{\varphi}_{20}-\overline{\varphi}_{10})-h_1\omega+M_1^*=0,\\ \overline{\Phi}_{20}=\omega^2L_5^*\sin(\overline{\varphi}_{20}-\overline{\varphi}_{10})-h_2\omega+M_2^*=0.\end{cases} \qquad (56)$$

From (56) the equality for determination of the frequency ω is obtained:

$$(h_1\omega-M_1^*)L_5^*-(h_2\omega-M_2^*)L_1^*=0, \qquad (57)$$

moreover, one is to take into account that $\omega^2\in\left(\frac{4c_0^2}{EF\cdot\rho F},+\infty\right)$.

The expressions for determination of the values $\overline{\varphi}_{20}-\overline{\varphi}_{10}$ are found:

$$\sin(\overline{\varphi}_{20}-\overline{\varphi}_{10})=\frac{h_1\omega-M_1^*}{\omega^2L_1^*}=\frac{h_2\omega-M_2^*}{\omega^2L_5^*}, \qquad (58)$$

hence it follows that the inequality

$$\left|\frac{h_1\omega-M_1^*}{\omega^2L_1^*}\right|=\left|\frac{h_2\omega-M_2^*}{\omega^2L_5^*}\right|<1 \qquad (59)$$

is the condition of existence of the steady-state regimes of motion.

The stability condition of the obtained regimes of motion is determined by the inequality

$$(L_5^*-L_1^*)\cos(\overline{\varphi}_{20}-\overline{\varphi}_{10})<0. \qquad (60)$$

Let $L_5^*\neq L_1^*\neq 0$, and $h_1\omega-M_1^*=h_2\omega-M_2^*=0$, then $\sin(\overline{\varphi}_{20}-\overline{\varphi}_{10})=0$, from where it is found $\overline{\varphi}_{20}-\overline{\varphi}_{10}=0;\pi$.

The solution $\overline{\varphi}_{20}-\overline{\varphi}_{10}=0$, according to (60), is stable when $L_5^*<L_1^*$, and $\overline{\varphi}_{20}-\overline{\varphi}_{10}=\pi$ is stable when $L_5^*>L_1^*$.

In the case, when $h_1\omega-M_1^*=h_2\omega-M_2^*\neq 0$ and $L_5^*=L_1^*\neq 0$, then $\sin(\overline{\varphi}_{20}-\overline{\varphi}_{10})=\frac{h_1\omega_0-M_1^*}{\omega_0^2L_1^*(\omega_0)}$, where ω_0 - the solution of the equation $L_5^*=L_1^*$. If the condition of existence is satisfied, i. e. $\left|\frac{h_1\omega_0-M_1^*}{\omega_0^2L_1^*(\omega_0)}\right|<1$, then the solution $\overline{\varphi}_{20}-\overline{\varphi}_{10}$ will be stable when the inequality $L_1^*\cos(\overline{\varphi}_{20}-\overline{\varphi}_{10})<0$ is stisfied: if the solution $\overline{\varphi}_{20}-\overline{\varphi}_{10}\in(-\pi/2,\pi/2)$, then it is stable when $L_1^*<0$, if $\overline{\varphi}_{20}-\overline{\varphi}_{10}\in(\pi/2,3\pi/2)$, then it will be stable when $L_1^*>0$.

In the case, when $h_1\omega-M_1^*\neq h_2\omega-M_2^*$ and $L_5^*\neq L_1^*\neq 0$, those solutions $\overline{\varphi}_{20}-\overline{\varphi}_{10}$ (58), satisfying the condition of existence, will be stable, which belong to

1) $(0,\pi/2)$, if $L_5^*<L_1^*$ and

$$\frac{h_1\omega-M_1^*}{\omega^2L_1^*}=\frac{h_2\omega-M_2^*}{\omega^2L_5^*}>0, \qquad (61)$$

2) $(\pi/2,\pi)$, if the inequalities $L_5^*>L_1^*$ and (61) are satisfied,

3) $(\pi,3\pi/2)$, if $L_5^*>L_1^*$ and

$$\frac{h_1\omega - M_1^*}{\omega^2 L_1^*} = \frac{h_2\omega - M_2^*}{\omega^2 L_5^*} < 0, \tag{62}$$

4) $(3\pi/2, 2\pi)$, if the inequalities $L_5^* < L_1^*$ and (62) are satisfied.

In all the analyzed cases $\omega^2 \in \left(\frac{4c_0^2}{EF \cdot \rho F}, +\infty\right)$.

Note: When $\eta_2 > 0$ and $\omega^2 \in \left(\frac{4c_0^2}{EF \cdot \rho F}, +\infty\right)$:

$$1 < e^{\frac{c_0 r}{EF}\eta_2} < e^{\left(\frac{k}{2}-\alpha\right)\eta_2} < e^{\frac{2c_0 r}{EF}\eta_2},\ 0 < e^{-\left(\frac{k}{2}+\alpha\right)\eta_2} < e^{-\frac{2c_0 r}{EF}\eta_2} < 1.$$

$L_5^* e^{-\frac{k}{2}\eta_2}$ with the growth of the degree $\left(\frac{k}{2}+\alpha\right)\eta_2$ approaches zero, i. e. beginning from a certain value of the frequency ω the value of the function $L_5^* e^{-\frac{k}{2}\eta_2}$ will differ by a small amount from zero.

7. The results of numerical calculations

As initial data for the case $\xi = 0$ it is accepted:

$a_1 = 2$, $b_1 = 2$, $\rho_1 = 2$, $n = 2$, $h_1 = 0.3$, $h_2 = -0.6$, $M_1^* = 0.5$, $M_2^* = -0.2$, $\eta_1 = 0.01$, $\eta_2 = 0.05$,

where $a_1 = \frac{m_0 + 2m}{\rho F \cdot r}$, $b_1 = \frac{2m}{\rho F}$, $\rho_1 = \sqrt{\frac{\rho}{E}} \cdot r$, $n = \frac{EF}{c_0 \cdot r}$, $h_j = \frac{H_{\varphi j}}{mr}$, $M_j^* = \frac{M_{\varphi j}}{mr}$, $\eta_j = \frac{x_j}{r}$.

In this case $\omega^2 = \frac{4c_0^2}{EF \cdot \rho F} = \frac{1}{4}$ or $\omega = 1/2$.

The equation for determination of the frequency ω has two solutions: $\omega_1 \approx 1.3952$, $\omega_2 \approx 10.0560$. Frequency ω_2 does not satisfy the condition of existence.

Since $\omega_1 > 1/2$, thus the values $\overline{\varphi}_{20} - \overline{\varphi}_{10}$ are found from the equation (58).

When $\omega_1 \approx 1.3952$, then $\sin(\overline{\varphi}_{20} - \overline{\varphi}_{10}) \approx -0.6967$, from where $\overline{\varphi}_{20} - \overline{\varphi}_{10} \approx -0.7708$ and $\overline{\varphi}_{20} - \overline{\varphi}_{10} \approx -2.3416$. $L_1^* < L_5^*$, consequently, the value $\overline{\varphi}_{20} - \overline{\varphi}_{10} \approx -2.3416$ corresponds to the stable solution.

At the point $\eta_1 = 0.01$ the function $u_{10} + x_0$ is defined, at the point $\eta_2 = 0.05$ - the function $u_{20} + x_0$. The dependence of the presented functions on t is represented in Fig. 2.

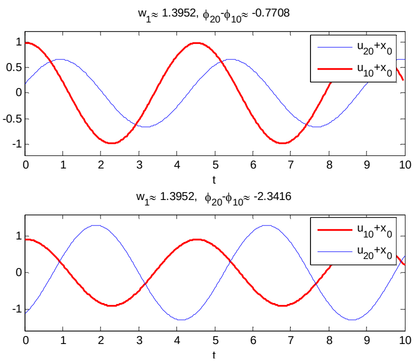

Fig. 2. The dependence of functions $u_{10} + x_0$ and $u_{20} + x_0$ on t (unstable and stable solution)

The three-dimensional image of the function

$$u(\eta,t) = \begin{cases} u^{(1)}(\eta,t),\ 0 \le \eta \le \eta_1, \\ u^{(2)}(\eta,t),\ \eta_1 \le \eta \le \eta_1 + \eta_2, \\ u^{(3)}(\eta,t),\ \eta \ge \eta_1 + \eta_2, \end{cases} \tag{63}$$

$\eta_1 = 0.01$, $\eta_2 = 0.05$, with the given parameters (in the case $\omega^2 > \frac{4c_0^2}{EF \cdot \rho F}$) is represented in Fig. 3.

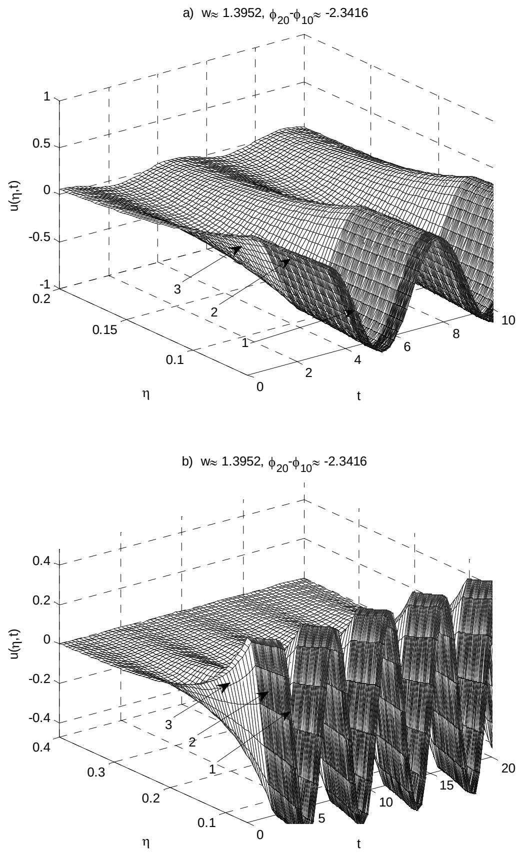

Fig. 3. The image of the function $u(\eta,t)$: $1 - u^{(1)}(\eta,t)$, $0 \le \eta \le 0.01$; $2 - u^{(2)}(\eta,t)$, $0.01 \le \eta \le 0.06$; $3 - u^{(3)}(\eta,t)$, $\eta \ge 0.06$

If to accept as initial the following data in the case $\xi = 0$:

$a_1 = 2$, $b_1 = 2$, $\rho_1 = 0.2$, $n = 2$, $h_1 = 0.2$, $h_2 = 0.7$, $M_1^* = M_2^* = 0.1$, $\eta_1 = 1$, $\eta_2 = 5$,

then two values of frequency will be obtained $\omega_1 \approx 1.47113$ and $\omega_2 \approx 1.71277$.

As $\omega^2 = \dfrac{4c_0^2}{EF \cdot \rho F} = 25$ or $\omega = 5$, and $\omega_1 < 5$ and $\omega_2 < 5$, then the values $\overline{\varphi}_{20} - \overline{\varphi}_{10}$ are found from the equation (50).

With ω_1 the equation (50) has solutions $\overline{\varphi}_{20} - \overline{\varphi}_{10} \approx 1{,}4379$ and $\overline{\varphi}_{20} - \overline{\varphi}_{10} \approx 3.0087$, and only the solution $\overline{\varphi}_{20} - \overline{\varphi}_{10} \approx 3.0087$ satisfies the stability condition (53).

With ω_2 from the two solutions $\overline{\varphi}_{20} - \overline{\varphi}_{10} \approx 1.3865$ and $\overline{\varphi}_{20} - \overline{\varphi}_{10} \approx 2.9573$ stable position will be the second one.

The dependence of functions $u_{10} + x_0$ ($\eta_1 = 1$) and $u_{20} + x_0$ ($\eta_2 = 5$) from t is represented in Fig. 4.

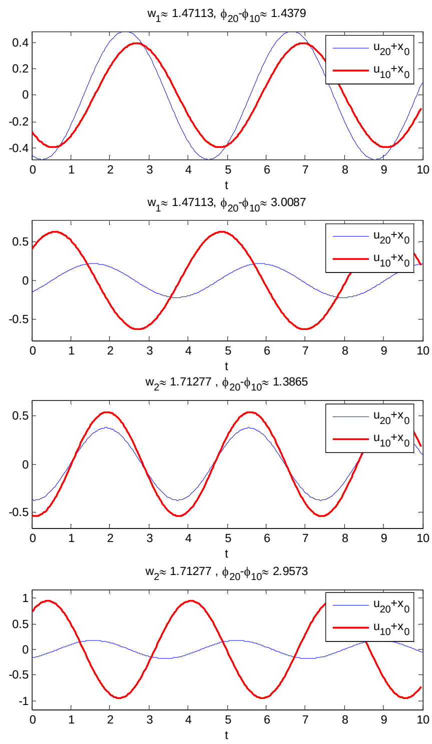

Fig. 4. Dependence of functions $u_{10} + x_0$ and $u_{20} + x_0$ from t (unstable and stable solution)

The three-dimensional image of the function $u(\eta, t)$ (63) $\eta_1 = 1$, $\eta_2 = 5$, with the given parameters (in the case $\omega^2 < \dfrac{4c_0^2}{EF \cdot \rho F}$) is represented in Fig. 5.

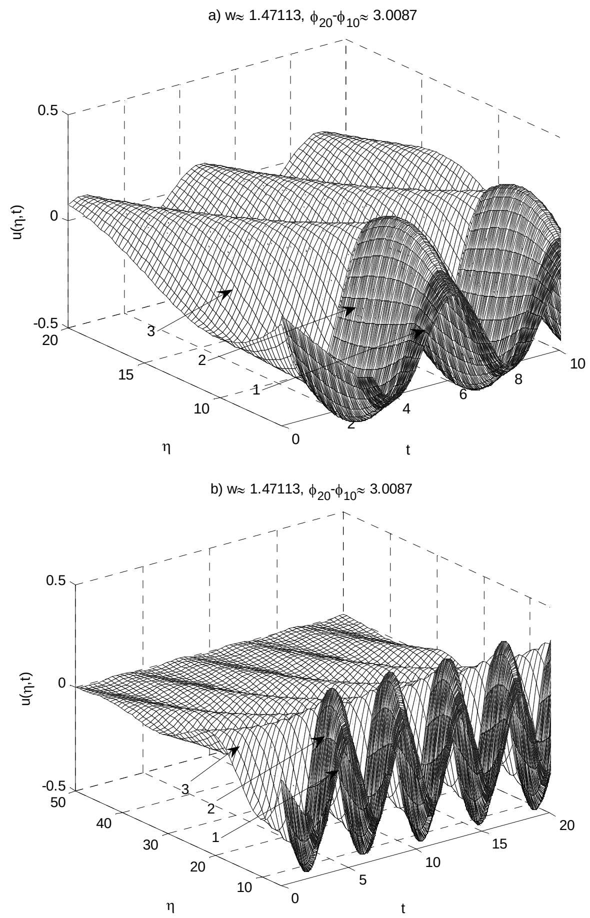

Fig. 5. Image of the function $u(\eta, t)$: $1 - u^{(1)}(\eta, t)$, $0 \le \eta \le 1$; $2 - u^{(2)}(\eta, t)$, $1 \le \eta \le 6$; $3 - u^{(3)}(\eta, t)$, $\eta \ge 6$

8. Conclusions

Approximate analytical solution of the examined problem about the synchronization of vibroexciters for the generation of longitudinal travelling waves in a bar can be disseminated to the more general case, for any number of vibration exciters in the system and for more complicated systems.

On the basis of results of investigations some qualities of the system are revealed, the obtained inequalities and equations are suitable for practical application.

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