# 525. Comparison of different strategies of integration of vibrating systems

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*(Received 30 September 2009; accepted 27 November 2009)*

**Abstract.** Different schemes of numerical integration of the second order differential equation using finite elements in time with linear interpolation are compared. The precision of integration is investigated and the recommendations for the choice of the time step are provided. Precision of the higher order three node numerical integration schemes with the displacement and velocity as the nodal variables is investigated and the recommendations for the choice of the time step are provided.

**Keywords:** one degree of freedom, vibrations, numerical integration, finite elements.

## Introduction

Numerical integration of equations is important for the analysis of a number of vibrating systems [1, 2, 3, 4]. The use of finite elements in time for performing numerical integration is described in [5, 6, 7, 8]. The comparison of various numerical integration schemes for the second order differential equations is performed in [9]. Different schemes of numerical integration of the second order differential equation using finite elements in time with linear interpolation are compared. The precision of integration is investigated and the recommendations for the choice of the time step are provided.

The precision of the higher order three node numerical integration scheme with the displacement and velocity as the nodal variables is investigated and the recommendations for the choice of the time step are provided. Also similar numerical integration scheme with quadratic interpolation is investigated.

Application of the generalized Galerkin procedure for the numerical integration using finite elements in time with linear interpolation is investigated.

## Integration with linear interpolation

The dynamics of a single degree of freedom vibrating system is described by the equation:

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f, \tag{1}$$

where $m$, $c$, $k$ and $f$ are the mass, damping, stiffness and force; $x$ is the displacement; $t$ is the time variable.

The displacement is interpolated as:

$$x = [N]\begin{Bmatrix} x_0 \\ x_T \end{Bmatrix}, \tag{2}$$

where the subscript denotes the value of the time variable ($T$ is the time step), while:

$$[N] = [N_1 \quad N_2], \tag{3}$$

where:

$$N_1 = 1 - \frac{t}{T}, \quad N_2 = \frac{t}{T}. \tag{4}$$

On the basis of the method of Galerkin:

$$\int_0^T [N]^T \left( m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx - f \right) dt = 0. \tag{5}$$

This gives:

$$\begin{aligned} &\left[ -m\int_0^T \left[\frac{dN}{dt}\right]^T \left[\frac{dN}{dt}\right] dt + c\int_0^T [N]^T \left[\frac{dN}{dt}\right] dt + k\int_0^T [N]^T [N] dt \right] \begin{Bmatrix} x_0 \\ x_T \end{Bmatrix} = \\ &= \left\{ f\int_0^T [N]^T dt \right\} + \begin{Bmatrix} m\dfrac{dx_0}{dt} \\ -m\dfrac{dx_T}{dt} \end{Bmatrix}. \end{aligned} \tag{6}$$

After performing the integrations:

$$\left[ -m\begin{bmatrix} \dfrac{1}{T} & -\dfrac{1}{T} \\ -\dfrac{1}{T} & \dfrac{1}{T} \end{bmatrix} + c\begin{bmatrix} -\dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix} + k\begin{bmatrix} \dfrac{T}{3} & \dfrac{T}{6} \\ \dfrac{T}{6} & \dfrac{T}{3} \end{bmatrix} \right] \begin{Bmatrix} x_0 \\ x_T \end{Bmatrix} = f\begin{Bmatrix} \dfrac{T}{2} \\ \dfrac{T}{2} \end{Bmatrix} + m\begin{Bmatrix} \dfrac{dx_0}{dt} \\ -\dfrac{dx_T}{dt} \end{Bmatrix}. \tag{7}$$

So the numerical integration is performed on the basis of the following equations:

$$\begin{aligned} &\left( m\frac{1}{T} + c\frac{1}{2} + k\frac{T}{6} \right) x_T = f\frac{T}{2} - \left( -m\frac{1}{T} - c\frac{1}{2} + k\frac{T}{3} \right) x_0 + m\frac{dx_0}{dt}, \\ &m\frac{dx_T}{dt} = f\frac{T}{2} - \left( m\frac{1}{T} - c\frac{1}{2} + k\frac{T}{6} \right) x_0 - \left( -m\frac{1}{T} + c\frac{1}{2} + k\frac{T}{3} \right) x_T. \end{aligned} \tag{8}$$

**Investigation of precision of integration with linear interpolation**

The following equation is analyzed:

$$\frac{d^2x}{dt^2} + x = 0, \tag{9}$$

with the initial conditions:

$$x\big|_{t=0} = 1, \quad \frac{dx}{dt}\bigg|_{t=0} = 0. \tag{10}$$

The analytical and numerical time histories of motion when 14 steps in the period of oscillations are used are presented in Fig. 1. The analytical time history is grey, while the numerical one is black.

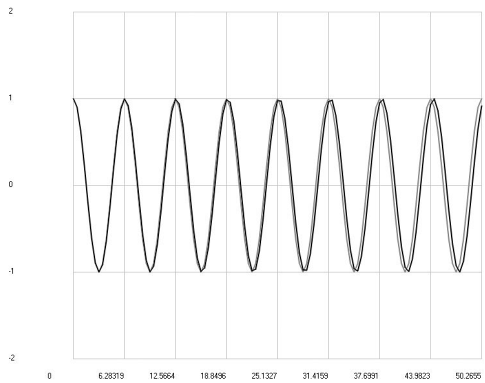


**Fig. 1.** Analytical and numerical time histories of motion (the analytical time history is grey, while the numerical one is black)

The value of the displacement after one period of oscillations as a function of the number of time steps is presented in Fig. 2. The value of the velocity after one period of oscillations as a function of the number of time steps is presented in Fig. 3.

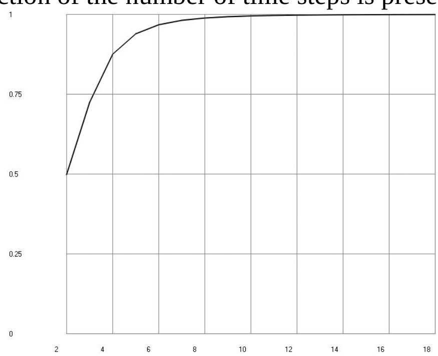


**Fig. 2.** Displacement after one period of oscillations as a function of the number of time steps

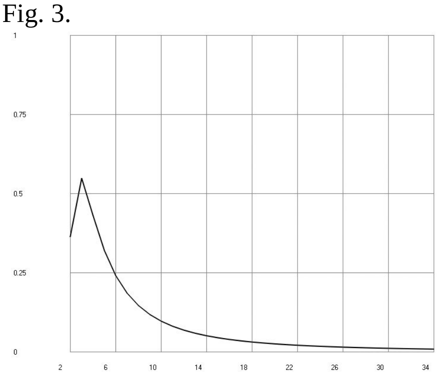


**Fig. 3.** Velocity after one period of oscillations as a function of the number of time steps

## Integration with higher order interpolation

The nodes of the finite element correspond to the values -1, 0 and 1 of the local coordinate $\xi$. Thus the shape functions are:

$$N_1 = \frac{1}{4}\left(4\xi^2 - 5\xi^3 - 2\xi^4 + 3\xi^5\right), \quad N_2 = \frac{1}{4}\left(\xi^2 - \xi^3 - \xi^4 + \xi^5\right),$$

$$N_3 = 1 - 2\xi^2 + \xi^4, \quad N_4 = \xi - 2\xi^3 + \xi^5,$$

$$N_5 = \frac{1}{4}\left(4\xi^2 + 5\xi^3 - 2\xi^4 - 3\xi^5\right), \quad N_6 = \frac{1}{4}\left(-\xi^2 - \xi^3 + \xi^4 + \xi^5\right). \tag{11}$$

So the numerical integration is performed on the basis of the following matrix equation of second order:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_T \\ \dfrac{dx_T}{dt} \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}, \tag{12}$$

where:

$$a_{11} = \frac{m}{T}\int_{-1}^{1} N_5 \frac{d^2 N_5}{d\xi^2} d\xi + c\int_{-1}^{1} N_5 \frac{dN_5}{d\xi} d\xi + kT\int_{-1}^{1} N_5 N_5 d\xi,$$

$$a_{21} = \frac{m}{T}\int_{-1}^{1} N_6 \frac{d^2 N_5}{d\xi^2} d\xi + c\int_{-1}^{1} N_6 \frac{dN_5}{d\xi} d\xi + kT\int_{-1}^{1} N_6 N_5 d\xi,$$

$$a_{12} = m\int_{-1}^{1} N_5 \frac{d^2 N_6}{d\xi^2} d\xi + cT\int_{-1}^{1} N_5 \frac{dN_6}{d\xi} d\xi + kT^2\int_{-1}^{1} N_5 N_6 d\xi,$$

$$a_{22} = m\int_{-1}^{1} N_6 \frac{d^2 N_6}{d\xi^2} d\xi + cT\int_{-1}^{1} N_6 \frac{dN_6}{d\xi} d\xi + kT^2\int_{-1}^{1} N_6 N_6 d\xi,$$

$$b_1 = T\int_{-1}^{1} N_5 d\xi f - \left(\frac{m}{T}\int_{-1}^{1} N_5 \frac{d^2 N_1}{d\xi^2} d\xi + c\int_{-1}^{1} N_5 \frac{dN_1}{d\xi} d\xi + kT\int_{-1}^{1} N_5 N_1 d\xi\right) x_{-T} -$$
$$-\left(m\int_{-1}^{1} N_5 \frac{d^2 N_2}{d\xi^2} d\xi + cT\int_{-1}^{1} N_5 \frac{dN_2}{d\xi} d\xi + kT^2\int_{-1}^{1} N_5 N_2 d\xi\right) \frac{dx_{-T}}{dt} -$$
$$-\left(\frac{m}{T}\int_{-1}^{1} N_5 \frac{d^2 N_3}{d\xi^2} d\xi + c\int_{-1}^{1} N_5 \frac{dN_3}{d\xi} d\xi + kT\int_{-1}^{1} N_5 N_3 d\xi\right) x_0 -$$
$$-\left(m\int_{-1}^{1} N_5 \frac{d^2 N_4}{d\xi^2} d\xi + cT\int_{-1}^{1} N_5 \frac{dN_4}{d\xi} d\xi + kT^2\int_{-1}^{1} N_5 N_4 d\xi\right) \frac{dx_0}{dt},$$

$$\begin{aligned} b_2 = T\int_{-1}^{1} N_6 d\xi f - \left( \frac{m}{T}\int_{-1}^{1} N_6 \frac{d^2N_1}{d\xi^2} d\xi + c\int_{-1}^{1} N_6 \frac{dN_1}{d\xi} d\xi + kT\int_{-1}^{1} N_6 N_1 d\xi \right) x_{-T} - \\ - \left( m\int_{-1}^{1} N_6 \frac{d^2N_2}{d\xi^2} d\xi + cT\int_{-1}^{1} N_6 \frac{dN_2}{d\xi} d\xi + kT^2\int_{-1}^{1} N_6 N_2 d\xi \right) \frac{dx_{-T}}{dt} - \\ - \left( \frac{m}{T}\int_{-1}^{1} N_6 \frac{d^2N_3}{d\xi^2} d\xi + c\int_{-1}^{1} N_6 \frac{dN_3}{d\xi} d\xi + kT\int_{-1}^{1} N_6 N_3 d\xi \right) x_0 - \\ - \left( m\int_{-1}^{1} N_6 \frac{d^2N_4}{d\xi^2} d\xi + cT\int_{-1}^{1} N_6 \frac{dN_4}{d\xi} d\xi + kT^2\int_{-1}^{1} N_6 N_4 d\xi \right) \frac{dx_0}{dt}. \end{aligned} \tag{13}$$

## Investigation of precision of integration with higher order interpolation

The analytical and numerical time histories of motion when 4 steps in the period of oscillations are used are presented in Fig. 4. The analytical time history is grey, while the numerical one is black.

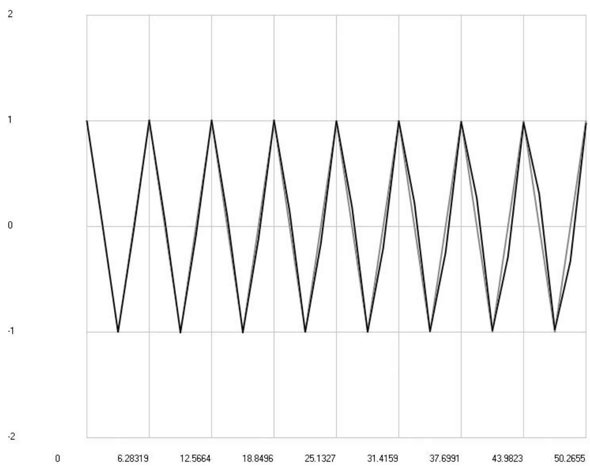


**Fig. 4.** Analytical and numerical time histories of motion (the analytical time history is grey, while the numerical one is black)

The value of the displacement after one period of oscillations as a function of the number of time steps is presented in Fig. 5. The value of the velocity after one period of oscillations as a function of the number of time steps is presented in Fig. 6.

## Integration with quadratic interpolation

The nodes of the finite element correspond to the values -1, 0 and 1 of the local coordinate $\xi$. Numerical integration is performed on the basis of the following equations:

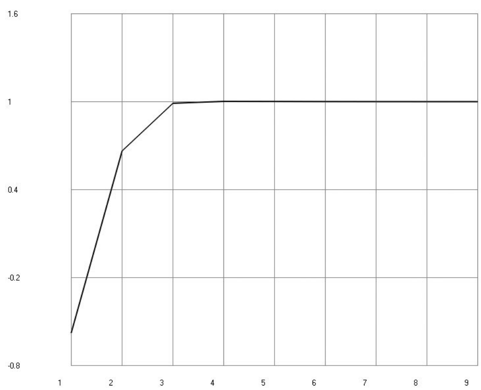


**Fig. 5.** Displacement after one period of oscillations as a function of the number of time steps

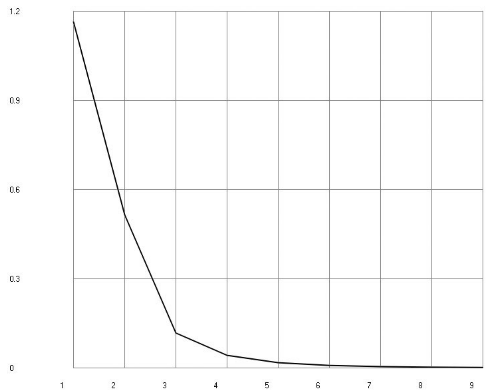


**Fig. 6.** Velocity after one period of oscillations as a function of the number of time steps

$$
\begin{aligned}
&\left(-\frac{m}{T}\int_{-1}^{1}\frac{dR_1}{d\xi}\frac{dN_3}{d\xi}d\xi + c\int_{-1}^{1}R_1\frac{dN_3}{d\xi}d\xi + kT\int_{-1}^{1}R_1N_3d\xi\right)x_T = T\int_{-1}^{1}R_1d\xi f - \\
&-\left(-\frac{m}{T}\int_{-1}^{1}\frac{dR_1}{d\xi}\frac{dN_1}{d\xi}d\xi + c\int_{-1}^{1}R_1\frac{dN_1}{d\xi}d\xi + kT\int_{-1}^{1}R_1N_1d\xi\right)x_{-T} - \\
&-\left(-\frac{m}{T}\int_{-1}^{1}\frac{dR_1}{d\xi}\frac{dN_2}{d\xi}d\xi + c\int_{-1}^{1}R_1\frac{dN_2}{d\xi}d\xi + kT\int_{-1}^{1}R_1N_2d\xi\right)x_0 + m\frac{dx_{-T}}{dt}, \\
&m\frac{dx_T}{dt} = T\int_{-1}^{1}R_2d\xi f - \left(-\frac{m}{T}\int_{-1}^{1}\frac{dR_2}{d\xi}\frac{dN_1}{d\xi}d\xi + c\int_{-1}^{1}R_2\frac{dN_1}{d\xi}d\xi + kT\int_{-1}^{1}R_2N_1d\xi\right)x_{-T} - \\
&-\left(-\frac{m}{T}\int_{-1}^{1}\frac{dR_2}{d\xi}\frac{dN_2}{d\xi}d\xi + c\int_{-1}^{1}R_2\frac{dN_2}{d\xi}d\xi + kT\int_{-1}^{1}R_2N_2d\xi\right)x_0 - \\
&-\left(-\frac{m}{T}\int_{-1}^{1}\frac{dR_2}{d\xi}\frac{dN_3}{d\xi}d\xi + c\int_{-1}^{1}R_2\frac{dN_3}{d\xi}d\xi + kT\int_{-1}^{1}R_2N_3d\xi\right)x_T,
\end{aligned}
\tag{14}
$$

where $N_i$ are quadratic shape functions and $R_i$ are linear shape functions.

## Investigation of precision of integration with quadratic interpolation

The values of the displacement and of the velocity after one period of oscillations as a function of the number of time steps are presented in Fig. 7.

$$\left(-\frac{m}{.5T}\int_{-1}^{1}\frac{dR_1}{d\xi}\frac{dN_2}{d\xi}d\xi+c\int_{-1}^{1}R_1\frac{dN_2}{d\xi}d\xi+k.5T\int_{-1}^{1}R_1N_2d\xi\right)x_T=.5T\int_{-1}^{1}R_1d\xi f-$$
$$-\left(-\frac{m}{.5T}\int_{-1}^{1}\frac{dR_1}{d\xi}\frac{dN_1}{d\xi}d\xi+c\int_{-1}^{1}R_1\frac{dN_1}{d\xi}d\xi+k.5T\int_{-1}^{1}R_1N_1d\xi\right)x_0+m\frac{dx_0}{dt},$$

$$m\frac{dx_T}{dt}=.5T\int_{-1}^{1}R_2d\xi f-\left(-\frac{m}{.5T}\int_{-1}^{1}\frac{dR_2}{d\xi}\frac{dN_1}{d\xi}d\xi+c\int_{-1}^{1}R_2\frac{dN_1}{d\xi}d\xi+k.5T\int_{-1}^{1}R_2N_1d\xi\right)x_0-$$
$$-\left(-\frac{m}{.5T}\int_{-1}^{1}\frac{dR_2}{d\xi}\frac{dN_2}{d\xi}d\xi+c\int_{-1}^{1}R_2\frac{dN_2}{d\xi}d\xi+k.5T\int_{-1}^{1}R_2N_2d\xi\right)x_T, \tag{15}$$

where $N_i$ are linear shape functions and $R_i = N_i$ for the Galerkin procedure, while for the generalized Galerkin procedure:

$$R_1=\cos^2\frac{\pi}{4}(1+\xi),\quad R_2=\sin^2\frac{\pi}{4}(1+\xi). \tag{16}$$

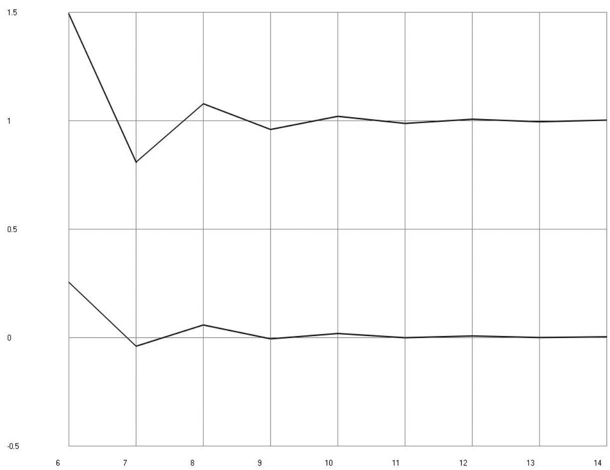


**Fig. 7.** Displacement and velocity after one period of oscillations as a function of the number of time steps

## Investigation of precision of integration with linear interpolation and special weighting functions

The values of the displacement and of the velocity after one period of oscillations as a function of the number of time steps are presented in Fig. 8. For the Galerkin procedure they are grey, while for the generalized Galerkin procedure they are black.

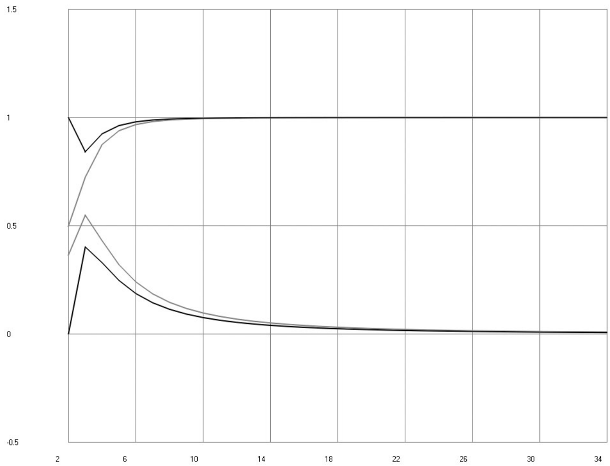


**Fig. 8.** Displacement and velocity after one period of oscillations as a function of the number of time steps (for the Galerkin procedure in grey, for the generalized Galerkin procedure in black)

## Conclusions

The numerical integration scheme for integration of the second order differential equation using finite elements in time with linear interpolation is proposed and investigated. On the basis of the obtained results the time step equal to about one fourteenth of the eigenperiod is recommended. The higher order three node numerical integration scheme with the displacement and velocity as the nodal variables is investigated. On the basis of the obtained results the time step equal to about one fourth of the eigenperiod is recommended.

Also similar numerical integration scheme with quadratic interpolation is investigated. Of course the precision of this integration procedure is lower than of the previous one, but it has the advantage of more simple calculations. Application of the generalized Galerkin procedure for the numerical integration using finite elements in time with linear interpolation is investigated. The choice of special weighting functions enables to increase the precision of integration.

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