# 550. Static and dynamic synthesis of partitioned substructures

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**Abstract.** Substructuring is to subdivide an overall structure into two or more substructures to reduce the model-order of the huge structural system. The problem to synthesize the substructures is established by a mathematical system consisting of equilibrium equations and prescribed compatibility conditions. Considering that the compatibility conditions are constraints, this study derives the analytical methods for describing the responses of constrained static and dynamic systems and provides a structural synthesis method based on the Guyan condensation method and the derived equations. The analysis process is carried out by partitioning into two regions of interior and boundary regions, and giving the compatibility conditions. And the dynamic analysis reduces model-order based on the constraint conditions between modal coordinates by the first several mode shape matrix. The validity of the proposed method is illustrated through the structural synthesis of stable and unstable substructures, and the structural reanalysis to evaluate the structural response for changes in the design without solving the complete set of modified simultaneous equations.

**Keywords:** substructures, compatibility, constraints, structural synthesis, equilibrium equation, mode shape.

## 1. Introduction

Substructuring includes a procedure that condenses a group of finite elements into one element represented as a matrix. The substructuring is to reduce computer time and to allow solution of very large problems with limited computer resources. A very large structural system is composed of substructures interconnected by springs and supports. The entire structure keeps the static equilibrium state by reactions and resists the externally provided vertical as well as lateral loads. Such structures include grid structures, longitudinal trusses, and slabs as diaphragms subjected to lateral forces, etc. The increase in the number of structural components yields a number of compatibility conditions and reaction forces to be determined, and requires more simplified analysis. Based on elastic analysis, the proper structural analysis depends on the determination of constraint forces at the interfaces of substructures to satisfy the given compatibility conditions.

There has been much research to consider structural synthesis of substructures. Substructure coupling methods are the techniques to reduce the model-order of huge structural systems. Hurty [1] introduced the component mode synthesis (CMS) method in 1960. The method is to combine subdivided substructures into an approximate mathematical model of the full structural system using the displacement constraints and the interface forces at the interfaces. A number of variants of the methods were proposed and employed [2-7].

Structural reanalysis refers to the analysis of a structure which has been slightly modified by the addition or deletion of structural members as substructures and is to evaluate the structural response for changes in the design without solving the complete set of modified simultaneous equations. The combined approximations approach to combine local and global approximations was developed for linear static reanalysis [8]. Kirsch [9] presented a general approach for structural optimization and the method integrated the constraint values and constraint derivatives into an effective optimization procedure. Kirsch and Papalambros [10] proposed a method not based on calculation of derivatives unlike common approximations of the structural response and the proposed method was illustrated in different types of design variables and structures. It is necessary to satisfy the compatibility conditions between substructures in combining the substructures into an entire structure. The CMS method has been derived based on the constraint conditions and the interface forces requirement, and leads to model-order reduction.

This study proposes static and dynamic methods to describe the constrained responses and to synthesize fixed-free or free-free end substructures into a complex structure in the satisfaction of the compatibility conditions. The synthesis approach is performed by partitioning into two regions of interior and boundary regions based on the Guyan condensation method [11] and the derived equations. All the DOFs except the interface are eliminated by means of static reduction to generate system matrices exclusively concerning the interface DOF. And the dynamic analysis reduces model-order based on the constraint conditions between modal coordinates by the first several mode shape matrix. The validity of the proposed method is illustrated through the structural synthesis of stable and unstable substructures, and the structural reanalysis to evaluate the structural response for changes in the design without solving the complete set of modified simultaneous equations.

## 2. Description of constrained responses

### 2.1 Constrained equation of static systems

Static responses of many practical structural systems are affected by the constraint conditions that include the support conditions, the compatibility conditions in structural systems and geometric requirements, etc. If constraints are given to a static system, the initial equilibrium equation must be modified to satisfy them.

The existence of the constraints needs to determine the displacement or force variations to be deviated from the initial state. The constraint forces prevent the system from deviations of the constrained manifold and are expressed by stiffness variation. The equilibrium equation of the constrained system is derived by combining the equilibrium equation of initial unconstrained system and the constraint equations.

Expressing that $\mathbf{K}_a$ is an $n \times n$ stiffness matrix of initial system, $\hat{\mathbf{u}}$ is an $n \times 1$ displacement vector, and F is an $n \times 1$ given force vector, the equilibrium equation of the initial system can be written by

$$\mathbf{K}_a \hat{\mathbf{u}} = \mathbf{F}. \tag{1}$$

The displacements of the system are calculated by

$$\hat{\mathbf{u}} = \mathbf{K}_a^{-1} \mathbf{F}, \tag{2}$$

where $\hat{\mathbf{u}}$ represents the initial displacement vector. And assume that the responses of the static system are restricted by m constraints

$$\mathbf{Au} = \mathbf{b}. \tag{3}$$

where A is an $m \times n (m < n)$ coefficient matrix, $\mathbf{b}$ indicates the $m \times 1$ vector and $\mathbf{u}$ is the $n \times 1$ actual displacement vector.

The displacements due to the existence of the constraints do not satisfy the equilibrium equation of Eqn. (1) and the equilibrium equation should be modified by the corrected stiffness matrix $\mathbf{K}$

$$\mathbf{Ku} = \mathbf{F}\,, \tag{4}$$

where we assume that the external force vector $\mathbf{F}$ is not changed.

The constraint force vector can be expressed as a function of the newly updated stiffness matrix. It can be obtained by minimizing a cost function in the satisfaction of the constraints. This study utilizes the cost function written as

$$J = \frac{1}{2}\left\|\mathbf{K}_a^{1/2}\left(\mathbf{K}^{-1} - \mathbf{K}^{-1}{}_a\right)\mathbf{K}_a^{1/2}\right\|. \tag{5}$$

Inserting Eqn. (4) into Eqn. (3), it can be written as

$$\mathbf{AK}^{-1}\mathbf{F} = \mathbf{b}\,. \tag{6}$$

Equation (6) is modified for minimizing the cost function as

$$\mathbf{AK}_a^{-1/2}\mathbf{K}_a^{1/2}\mathbf{K}^{-1}\mathbf{K}_a^{1/2}\mathbf{K}_a^{-1/2}\mathbf{F} = \mathbf{b}\,. \tag{7}$$

Letting $\mathbf{R} = \mathbf{AK}_a^{-1/2}$ which is an $m \times n$ rectangular matrix and $\mathbf{D} = \mathbf{K}_a^{-1/2}\mathbf{F}$, and solving Eqn. (7) with respect to $\mathbf{K}_a^{1/2}\mathbf{K}^{-1}\mathbf{K}_a^{1/2}\mathbf{D}$ based on the generalized solution of Moore-Penrose inverse [12], we obtain that

$$\mathbf{K}_a^{1/2}\mathbf{K}^{-1}\mathbf{K}_a^{1/2}\mathbf{D} = \mathbf{R}^{+}\mathbf{b} + \left(\mathbf{I} - \mathbf{R}^{+}\mathbf{R}\right)\mathbf{y}\,, \tag{8}$$

where y is an arbitrary vector, '+' denotes the Moore-Penrose inverse and I is an identity matrix. Inserting the condition to minimize the cost function of Eqn. (5) into Eqn. (8) and solving the result with respect to the arbitrary vector, we obtain that

$$\mathbf{y} = \left(\mathbf{I} - \mathbf{R}^{+}\mathbf{R}\right)\left(\mathbf{D} - \mathbf{R}^{+}\mathbf{b}\right) + \mathbf{R}^{+}\mathbf{Rz}\,, \tag{9}$$

where z is an arbitrary vector. Substituting Eqn. (9) into Eqn. (8), it can be written as

$$\mathbf{K}_a^{1/2}\mathbf{K}^{-1}\mathbf{K}_a^{1/2}\mathbf{D} = \mathbf{R}^{+}\mathbf{b} + \mathbf{D} - \mathbf{R}^{+}\mathbf{RD}\,. \tag{10}$$

Again, solving Eqn. (10) with respect to $\mathbf{K}_a^{1/2}\mathbf{K}^{-1}\mathbf{K}_a^{1/2}$, it follows that

$$\mathbf{K}_a^{1/2}\mathbf{K}^{-1}\mathbf{K}_a^{1/2} = \left(\mathbf{R}^{+}\mathbf{b} + \mathbf{D} - \mathbf{R}^{+}\mathbf{RD}\right)\mathbf{D}^{+} + \mathbf{h}\left(\mathbf{I} - \mathbf{DD}^{+}\right), \tag{11}$$

where h is an arbitrary matrix. Using the condition to minimize the cost function of Eqn. (5) into Eqn. (11) and solving the result with respect to the arbitrary matrix, we obtain that

$$\mathbf{h} = \left[\mathbf{I} - \left(\mathbf{R}^{+}\mathbf{b} + \mathbf{D} - \mathbf{R}^{+}\mathbf{RD}\right)\mathbf{D}^{+}\right]\left(\mathbf{I} - \mathbf{DD}^{+}\right) + \mathbf{sDD}^{+} = \mathbf{I} - \mathbf{DD}^{+} + \mathbf{sDD}^{+}\,, \tag{12}$$

where s is an arbitrary matrix. The substitution of Eqn. (12) into Eqn. (11) results in

$$\mathbf{K}_a^{1/2}\mathbf{K}^{-1}\mathbf{K}_a^{1/2} = \left(\mathbf{R}^{+}\mathbf{b} - \mathbf{R}^{+}\mathbf{RD}\right)\mathbf{D}^{+} + \mathbf{I}\,. \tag{13}$$

Premultiplying and postmultiplying both sides of Eqn. (13) by $\mathbf{K}_a^{-1/2}$, the inverse of the updated stiffness matrix is derived as

$$\mathbf{K}^{-1} = \mathbf{K}_a^{-1} + \mathbf{K}_a^{-1/2}\left(\mathbf{AK}_a^{-1/2}\right)^{+}\left(\mathbf{b} - \mathbf{A\hat{u}}\right)\left(\mathbf{K}_a^{-1/2}\mathbf{F}\right)^{+}\mathbf{K}_a^{-1/2}\,. \tag{14}$$

Equation (14) represents the inverse matrix of the corrected stiffness matrix due to the existence of the constraints and incorporates the constrained effects. Substituting Eqn. (14) into $\mathbf{u} = \mathbf{K}^{-1}\mathbf{F}$

with the property of $\left(\mathbf{K}_a^{-1/2}\mathbf{F}\right)^{+}\mathbf{K}_a^{-1/2}\mathbf{F} = 1$, the equilibrium equation of constrained static system can be derived

$$\mathbf{u} = \hat{\mathbf{u}} + \mathbf{K}_a^{-1/2}\left(\mathbf{A}\mathbf{K}_a^{-1/2}\right)^{+}\left(\mathbf{b} - \mathbf{A}\hat{\mathbf{u}}\right). \tag{15}$$

It is observed that this result corresponds with the one provided by Eun, Lee and Chung [13]. The second term of the right-hand side of Eqn. (15) denotes the displacement variation deviated from the initial state and the constraint force vector is obtained by premultiplying $\mathbf{K}_a$ on its second term as

$$\mathbf{F}^c = \mathbf{K}_a^{1/2}\left(\mathbf{A}\mathbf{K}_a^{-1/2}\right)^{+}\left(\mathbf{b} - \mathbf{A}\hat{\mathbf{u}}\right). \tag{16}$$

The derived equilibrium equation can be utilized in synthesizing partitioned substructures into an entire complex system by combining the equilibrium equations of all the substructures and the compatibility conditions.

## 2.2 Constrained dynamic systems

The constraint forces for dynamic systems are expressed by the mass variation of inertia force term unlike the static systems. Let us assume the stiffness and mass matrices of initial dynamic system to be $\mathbf{K}_a$ and $\mathbf{M}_a$, respectively. The dynamic responses of a system which is assumed to be linear and approximately discretized for n degrees of freedom (DOFs) can be described by

$$\mathbf{M}_a\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t), \tag{17}$$

where $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}^T$, $\mathbf{C} \in R^{n\times n}$ and $\mathbf{K} \in R^{n\times n}$ are the damping and stiffness matrices, respectively. Or the dynamic equations can be expressed in matrix form as

$$\mathbf{M}_a\ddot{\mathbf{q}} = \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, t), \tag{18}$$

where $\mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, t) = -\mathbf{C}\dot{\mathbf{q}} - \mathbf{K}\mathbf{q} + \mathbf{f}(t)$. The acceleration vector of unconstrained dynamic system, $\mathbf{a}(\mathbf{q}, \dot{\mathbf{q}}, t)$, can be written as

$$\mathbf{a} = \mathbf{M}_a^{-1}\mathbf{F}. \tag{19}$$

Let us assume that the system is constrained by $m$ constraint equations expressed as

$$\mathbf{A}(\mathbf{q}, \dot{\mathbf{q}}, t)\ddot{\mathbf{q}} = \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}, t), \tag{20}$$

where A is an $m\times n$ matrix and $\mathbf{b}$ is an $m\times 1$ vector. It is known that the dynamic responses of constrained system must satisfy the constraint equations at all times during numerical integration. The corrected mass matrix incorporates the constraint forces required for satisfying the constraints. The constrained dynamic equation due to the constraints such as Eqn. (20) is modified by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t), \tag{21}$$

where $\mathbf{M}$ denotes the corrected mass matrix.

The cost function for predicting the corrected mass matrix is written as

$$J = \frac{1}{2}\left\|\mathbf{M}_a^{1/2}\left(\mathbf{M}^{-1} - \mathbf{M}^{-1}{}_a\right)\mathbf{M}_a^{1/2}\right\|. \tag{22}$$

The corrected mass matrix can be obtained by minimizing the cost function of Eqn. (22). Utilizing Eqn. (21) into Eqn. (20), it follows that

$$\mathbf{AM}^{-1}\left(-\mathbf{C}\dot{\mathbf{q}}-\mathbf{Kq}+\mathbf{f}\right)=\mathbf{b}\,, \tag{23}$$

And the modification of Eqn. (23) leads to

$$\mathbf{AM}_a^{-1/2}\mathbf{M}_a^{1/2}\mathbf{M}^{-1}\mathbf{M}_a^{1/2}\mathbf{M}_a^{1/2}\mathbf{a}=\mathbf{b}\,, \tag{24}$$

where $\mathbf{a}$ represents the acceleration vector of unconstrained system of Eqn. (19). Letting $\mathbf{R}=\mathbf{AM}_a^{-1/2}$ and solving Eqn. (24) with respect to $\mathbf{M}_a^{1/2}\mathbf{M}^{-1}\mathbf{M}_a^{1/2}\mathbf{M}_a^{1/2}\mathbf{a}$, it follows that

$$\mathbf{M}_a^{1/2}\mathbf{M}^{-1}\mathbf{M}_a^{1/2}\mathbf{M}_a^{1/2}\mathbf{a}=\mathbf{R}^{+}\mathbf{b}+\left(\mathbf{I}-\mathbf{R}^{+}\mathbf{R}\right)\mathbf{y}\,, \tag{25}$$

where $\mathbf{y}$ is an arbitrary vector. Utilizing the condition to minimize Eqn. (22) into Eqn. (25), it satisfies

$$\mathbf{R}^{+}\mathbf{b}+\left(\mathbf{I}-\mathbf{R}^{+}\mathbf{R}\right)\mathbf{y}=\mathbf{M}_a^{1/2}\mathbf{a}\,. \tag{26}$$

Solving Eqn. (26) with respect to the arbitrary vector, we obtain that

$$\mathbf{y}=\left(\mathbf{I}-\mathbf{R}^{+}\mathbf{R}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}-\mathbf{R}^{+}\mathbf{b}\right)+\mathbf{R}^{+}\mathbf{Rz}\,, \tag{27}$$

where z is another arbitrary vector. The substitution of Eqn. (27) into Eqn. (25) yields

$$\mathbf{M}_a^{1/2}\mathbf{M}^{-1}\mathbf{M}_a^{1/2}\mathbf{M}_a^{1/2}\mathbf{a}=\mathbf{R}^{+}\mathbf{b}+\mathbf{M}_a^{1/2}\mathbf{a}-\mathbf{R}^{+}\mathbf{RM}_a^{1/2}\mathbf{a}\,. \tag{28}$$

Again, solving Eqn. (28) with respect to $\mathbf{M}_a^{1/2}\mathbf{M}^{-1}\mathbf{M}_a^{1/2}$, it can be written as

$$\mathbf{M}_a^{1/2}\mathbf{M}^{-1}\mathbf{M}_a^{1/2}=\left(\mathbf{R}^{+}\mathbf{b}+\mathbf{M}_a^{1/2}\mathbf{a}-\mathbf{R}^{+}\mathbf{RM}_a^{1/2}\mathbf{a}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}+\mathbf{r}\left[\mathbf{I}-\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}\right], \tag{29}$$

where r is an arbitrary matrix. Using the condition to minimize Eqn. (22) into Eqn. (29), it follows that

$$\left(\mathbf{R}^{+}\mathbf{b}+\mathbf{M}_a^{1/2}\mathbf{a}-\mathbf{R}^{+}\mathbf{RM}_a^{1/2}\mathbf{a}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}+\mathbf{r}\left[\mathbf{I}-\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}\right]=\mathbf{I}\,. \tag{30}$$

The unknown arbitrary matrix $\mathbf{r}$ can be obtained by solving Eqn. (30) as

$$\mathbf{r}=\mathbf{I}-\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}+\mathbf{d}\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}\,, \tag{31}$$

where d is an arbitrary matrix. Substituting Eqn. (31) into Eqn. (29) and arranging the result with $\left[\mathbf{I}-\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}\right]\left[\mathbf{I}-\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}\right]=\mathbf{I}-\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}$, we obtain that

$$\mathbf{M}_a^{1/2}\mathbf{M}^{-1}\mathbf{M}_a^{1/2}=\left(\mathbf{R}^{+}\mathbf{b}-\mathbf{R}^{+}\mathbf{RM}_a^{1/2}\mathbf{a}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}+\mathbf{I}\,. \tag{32}$$

Premultiplying and postmultiplying both sides of Eqn. (32) by $\mathbf{M}_a^{-1/2}$, the inverse of the corrected mass matrix can be written as

$$\mathbf{M}^{-1}=\mathbf{M}_a^{-1}+\mathbf{M}_a^{-1/2}\left(\mathbf{AM}_a^{-1/2}\right)^{+}\left(\mathbf{b}-\mathbf{Aa}\right)\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}\mathbf{M}_a^{-1/2}\,. \tag{33}$$

Inserting Eqn. (33) into Eqn. (21), the acceleration vector of the constrained system with the relation of $\left(\mathbf{M}_a^{1/2}\mathbf{a}\right)^{+}\mathbf{M}_a^{-1/2}\left(-\mathbf{C}\dot{\mathbf{q}}-\mathbf{Kq}+\mathbf{f}\right)=\mathbf{I}$ can be obtained as

$$\ddot{\mathbf{q}}=\mathbf{a}+\mathbf{M}_a^{-1/2}\left(\mathbf{AM}_a^{-1/2}\right)^{+}\left(\mathbf{b}-\mathbf{Aa}\right). \tag{34}$$

Premultiplying both sides of Eqn. (34) by $\mathbf{M}_a$, the second term of the result represents the constraint force vector as

$$\mathbf{F}^c = \mathbf{M}_a^{1/2}\left(\mathbf{A}\mathbf{M}_a^{-1/2}\right)^+\left(\mathbf{b}-\mathbf{A}\mathbf{a}\right). \tag{35}$$

The dynamic equation and the constraint force vector derived in this study exactly correspond with the ones provided by Udwadia and Kalaba [14] although the starting points are different. The equation can be utilized in combining subdivided substructures by constrained conditions into a huge dynamic system.

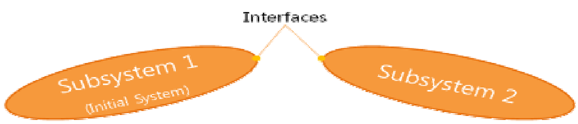
## 3. Synthesis of partitioned substructures

### 3.1 Static synthesis of substructures

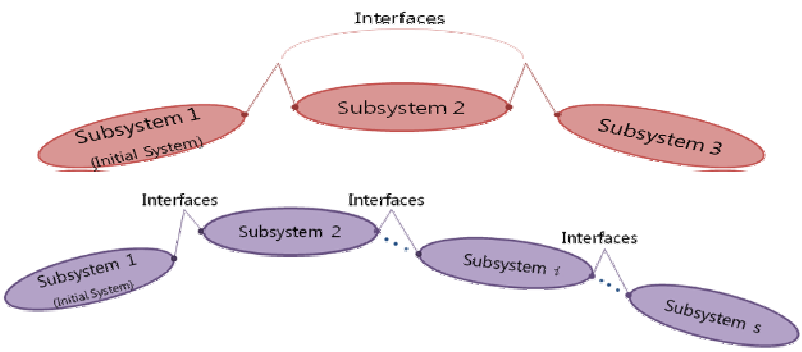
This section considers the analytical method to synthesize partitioned substructures into an entire complex structure based on the derived equation. Let us consider an initial system shown in Fig. 1(a). The static equilibrium equation of the initial system 1 described by an $n\times 1$ displacement vector $\mathbf{u}^{(1)}$ can be written as

$$\mathbf{K}^{(1)}\mathbf{u}^{(1)} = \mathbf{f}^{(1)} \tag{36}$$

where the superscript (1) denotes the initial system, $\mathbf{K}^{(1)}$ is an $n\times n$ positive-definite stiffness matrix and $\mathbf{f}^{(1)}$ is an $n\times 1$ force vector.




(a)




(b)

**Fig. 1.** Synthesis of substructures; (a) two substructures, (b) several substructures

Let us assume that a subsystem 2 is newly interconnected to the initial system shown in Fig. 1(a). Describing its static responses by $r\times 1$ displacement vector $\mathbf{u}^{(2)}$, its equilibrium equation is written as

$$\mathbf{K}^{(2)}\mathbf{u}^{(2)} = \mathbf{f}^{(2)}, \tag{37}$$

where the superscript (2) represents the subsystem interconnected to the initial system and $\mathbf{K}^{(2)}$ is an $(r \times r)$ positive-definite stiffness matrix. Let us partition the initial system and the subsystem into two regions of interior and boundary, respectively.

Expressing the partitioned displacements of the initial system as $\mathbf{u}^{(1)} = \left[\mathbf{u}_a^{(1)} \quad \mathbf{u}_b^{(1)}\right]^T$ where $\mathbf{u}_a^{(1)}$ denotes an $m \times 1$ interior displacement vector and $\mathbf{u}_b^{(1)}$ is an $(n-m) \times 1$ boundary displacement vector, the equilibrium equation can be written as

$$\begin{bmatrix} \mathbf{K}_{aa}^{(1)} & \mathbf{K}_{ab}^{(1)} \\ \mathbf{K}_{ba}^{(1)} & \mathbf{K}_{bb}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_a^{(1)} \\ \mathbf{u}_b^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_a^{(1)} \\ \mathbf{f}_b^{(1)} \end{bmatrix}. \tag{38}$$

Similarly, the equilibrium equation of the subsystem can be expressed as

$$\begin{bmatrix} \mathbf{K}_{aa}^{(2)} & \mathbf{K}_{ab}^{(2)} \\ \mathbf{K}_{ba}^{(2)} & \mathbf{K}_{bb}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_a^{(2)} \\ \mathbf{u}_b^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_a^{(2)} \\ \mathbf{f}_b^{(2)} \end{bmatrix}, \tag{39}$$

where $\mathbf{u}^{(2)} = \left[\mathbf{u}_a^{(2)} \quad \mathbf{u}_b^{(2)}\right]^T$ and $\mathbf{u}_a^{(2)}$ denotes $(n-m) \times 1$ boundary displacement vector and $\mathbf{u}_b^{(2)}$ represents $(r-n+m) \times 1$ interior displacement vector. Solving the first equation of Eqn. (38) with respect to $\mathbf{u}_a^{(1)}$, it can be expressed as

$$\mathbf{u}_a^{(1)} = -\mathbf{K}_{aa}^{(1)^{-1}} \left[\mathbf{K}_{ab}^{(1)} \mathbf{u}_b^{(1)} - \mathbf{f}_a^{(1)}\right] \tag{40}$$

And solving the second equation of Eqn. (38) with respect to $\mathbf{u}_b^{(1)}$, substituting Eqn. (40) into the result, and arranging it, we obtain that

$$\mathbf{K}_b^* \mathbf{u}_b^{(1)} = \mathbf{f}_b^*, \tag{41}$$

where $\mathbf{K}_b^* = \mathbf{I} - \mathbf{K}_{bb}^{(1)^{-1}} \mathbf{K}_{ba}^{(1)} \mathbf{K}_{aa}^{(1)^{-1}} \mathbf{K}_{ab}^{(1)}$ and $\mathbf{f}_b^* = \mathbf{K}_{bb}^{(1)^{-1}} \mathbf{f}_b^{(1)} - \mathbf{K}_{bb}^{(1)^{-1}} \mathbf{K}_{ba}^{(1)} \mathbf{K}_{aa}^{(1)^{-1}} \mathbf{f}_a^{(1)}$.
By the similar procedure, Eqn. (39) can be resolved as

$$\mathbf{u}_b^{(2)} = -\mathbf{K}_{bb}^{(2)^{-1}} \left[\mathbf{K}_{ba}^{(2)} \mathbf{u}_a^{(2)} - \mathbf{f}_b^{(2)}\right], \tag{42}$$

$$\mathbf{K}_a^* \mathbf{u}_a^{(2)} = \mathbf{f}_a^*, \tag{43}$$

where $\mathbf{K}_a^* = \mathbf{I} - \mathbf{K}_{aa}^{(2)^{-1}} \mathbf{K}_{ab}^{(2)} \mathbf{K}_{bb}^{(2)^{-1}} \mathbf{K}_{ba}^{(2)}$ and $\mathbf{f}_a^* = \mathbf{K}_{aa}^{(2)^{-1}} \mathbf{f}_a^{(2)} - \mathbf{K}_{aa}^{(2)^{-1}} \mathbf{K}_{ab}^{(2)} \mathbf{K}_{bb}^{(2)^{-1}} \mathbf{f}_b^{(2)}$.

The entire system must satisfy the compatibility conditions at the interfaces between the initial and added structures. The compatibility conditions at the $(n-m)$ interfaces can be written as

$$\mathbf{u}_b^{(1)} = \mathbf{u}_a^{(2)}. \tag{44}$$

Substituting the equilibrium equations of Eqns. (41) and (43), and the constraint equation of Eqn. (44) into Eqn. (15), and solving it, the static responses at the interfaces are calculated and the substitution of the results into Eqns. (40) and (42) leads to the displacements of the interior regions of the initial system and the subsystem. It is investigated that the structural analysis of the entire structure is performed by two stages on the boundary regions and the interior regions.

The proposed equation is extended to a generalized method to synthesize a series of substructures. Let us consider the structural synthesis of an entire system composed of an initial system and two or more subsystems shown in Fig. 1(b). And assume that the stiffness matrices of all substructures are full-rank. The equilibrium equations of the initial system and the $(s-1)$ subsystems can be written as

$$\begin{bmatrix} \mathbf{K}_{aa}^{(1)} & \mathbf{K}_{ab}^{(1)} \\ \mathbf{K}_{ba}^{(1)} & \mathbf{K}_{bb}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{a}^{(1)} \\ \mathbf{u}_{b}^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{a}^{(1)} \\ \mathbf{f}_{b}^{(1)} \end{bmatrix}, \tag{45a}$$

$$\begin{bmatrix} \mathbf{K}_{aa}^{(2)} & \mathbf{K}_{ab}^{(2)} & \mathbf{K}_{ac}^{(2)} \\ \mathbf{K}_{ba}^{(2)} & \mathbf{K}_{bb}^{(2)} & \mathbf{K}_{bc}^{(2)} \\ \mathbf{K}_{ca}^{(2)} & \mathbf{K}_{cb}^{(2)} & \mathbf{K}_{cc}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{a}^{(2)} \\ \mathbf{u}_{b}^{(2)} \\ \mathbf{u}_{c}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{a}^{(2)} \\ \mathbf{f}_{b}^{(2)} \\ \mathbf{f}_{c}^{(2)} \end{bmatrix} \cdots, \tag{45b}$$

$$\begin{bmatrix} \mathbf{K}_{aa}^{(i)} & \mathbf{K}_{ab}^{(i)} & \mathbf{K}_{ac}^{(i)} \\ \mathbf{K}_{ba}^{(i)} & \mathbf{K}_{bb}^{(i)} & \mathbf{K}_{bc}^{(i)} \\ \mathbf{K}_{ca}^{(i)} & \mathbf{K}_{cb}^{(i)} & \mathbf{K}_{cc}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{a}^{(i)} \\ \mathbf{u}_{b}^{(i)} \\ \mathbf{u}_{c}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{a}^{(i)} \\ \mathbf{f}_{b}^{(i)} \\ \mathbf{f}_{c}^{(i)} \end{bmatrix} \cdots, \tag{45c}$$

$$\begin{bmatrix} \mathbf{K}_{aa}^{(s)} & \mathbf{K}_{ab}^{(s)} \\ \mathbf{K}_{ba}^{(s)} & \mathbf{K}_{bb}^{(s)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{a}^{(s)} \\ \mathbf{u}_{b}^{(s)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{a}^{(s)} \\ \mathbf{f}_{b}^{(s)} \end{bmatrix}, \tag{45d}$$

where the superscript (1) denotes the initial system, the subscript 'b' in Eqn. (45a), the 'a' and 'c' in Eqns. (45b), (45c) and (45d) denote the boundary regions. And the subscript 'a' in Eqn. (45a) and 'b' in Eqn. (45d) represent the interior regions.

The displacements between two adjacent systems must satisfy the compatibility conditions. The $(s-1)$ compatibility conditions can be written as

$$\mathbf{u}_{b}^{(1)} = \mathbf{u}_{a}^{(2)}. \tag{46a}$$

$$\mathbf{u}_{c}^{(2)} = \mathbf{u}_{a}^{(3)} \cdots. \tag{46b}$$

$$\mathbf{u}_{c}^{(i-1)} = \mathbf{u}_{a}^{(i)} \cdots. \tag{46c}$$

$$\mathbf{u}_{c}^{(s-1)} = \mathbf{u}_{a}^{(s)}. \tag{46d}$$

The first and second equations of Eqn. (45a) for the initial system can be written as

$$\mathbf{K}_{aa}^{(1)}\mathbf{u}_{a}^{(1)} = -\left[\mathbf{K}_{ab}^{(1)}\mathbf{u}_{b}^{(1)} - \mathbf{f}_{a}^{(1)}\right], \tag{47}$$

$$\hat{\mathbf{K}}^{(1)}\mathbf{u}_{b}^{(1)} = \hat{\mathbf{f}}^{(1)}, \tag{48}$$

where $\hat{\mathbf{K}}^{(1)} = \mathbf{I} - \mathbf{K}_{bb}^{(1)^{-1}}\mathbf{K}_{ba}^{(1)}\mathbf{K}_{aa}^{(1)^{-1}}\mathbf{K}_{ab}^{(1)}$ and $\hat{\mathbf{f}}^{(1)} = \mathbf{K}_{bb}^{(1)^{-1}}\mathbf{f}_{b}^{(1)} - \mathbf{K}_{bb}^{(1)^{-1}}\mathbf{K}_{ba}^{(1)}\mathbf{K}_{aa}^{(1)^{-1}}\mathbf{f}_{a}^{(1)}$.
And Eqn. (45b) can be solved as

$$\mathbf{u}_{a}^{(2)} = -\mathbf{K}_{aa}^{(2)^{-1}}\left[\mathbf{K}_{ab}^{(2)}\mathbf{u}_{b}^{(2)} + \mathbf{K}_{ac}^{(2)}\mathbf{u}_{c}^{(2)} - \mathbf{f}_{a}^{(2)}\right], \tag{49a}$$

$$\mathbf{u}_b^{(2)} = -\mathbf{K}_{bb}^{(2)^{-1}}\left[\mathbf{K}_{ba}^{(2)}\mathbf{u}_a^{(2)} + \mathbf{K}_{bc}^{(2)}\mathbf{u}_c^{(2)} - \mathbf{f}_b^{(2)}\right], \tag{49b}$$

$$\mathbf{u}_c^{(2)} = -\mathbf{K}_{cc}^{(2)^{-1}}\left[\mathbf{K}_{ca}^{(2)}\mathbf{u}_a^{(2)} + \mathbf{K}_{cb}^{(2)}\mathbf{u}_b^{(2)} - \mathbf{f}_c^{(2)}\right]. \tag{49c}$$

Solving the three simultaneous equations of Eqns. (49) with respect to $\mathbf{u}_a^{(2)}$, $\mathbf{u}_b^{(2)}$ and $\mathbf{u}_c^{(2)}$, they can be derived as

$$\hat{\mathbf{K}}_a^{(2)}\mathbf{u}_a^{(2)} = \hat{\mathbf{f}}_a^{(2)}, \tag{50a}$$

$$\mathbf{K}_{bb}^{(2)}\mathbf{u}_b^{(2)} = \hat{\mathbf{f}}_b^{(2)}, \tag{50b}$$

$$\hat{\mathbf{K}}_c^{(2)}\mathbf{u}_c^{(2)} = \hat{\mathbf{K}}_{a2}^{(2)}\hat{\mathbf{f}}_c^{(2)} + \hat{\mathbf{f}}_c^{(2)}, \tag{50c}$$

where

$$\hat{\mathbf{K}}_a^{(2)} = \mathbf{K}_{aa}^{(2)^{-1}}\left(\mathbf{K}_{ab}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{K}_{bc}^{(2)} - \mathbf{K}_{ac}^{(2)}\right)\left(\mathbf{I} - \mathbf{K}_{cc}^{(2)^{-1}}\mathbf{K}_{cb}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{K}_{bc}^{(2)}\right)^{-1} \cdot \\ \left[\mathbf{K}_{cc}^{(2)^{-1}}\left(\mathbf{K}_{ca}^{(2)} - \mathbf{K}_{cb}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{K}_{ba}^{(2)}\right)\right] + \mathbf{I} - \mathbf{K}_{aa}^{(2)^{-1}}\mathbf{K}_{ab}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{K}_{ba}^{(2)},$$

$$\hat{\mathbf{f}}_a^{(2)} = -\mathbf{K}_{aa}^{(2)^{-1}}\left(\mathbf{K}_{ab}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{K}_{bc}^{(2)} - \mathbf{K}_{ac}^{(2)}\right)\left(\mathbf{I} - \mathbf{K}_{cc}^{(2)^{-1}}\mathbf{K}_{cb}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{K}_{bc}^{(2)}\right)^{-1} \cdot \\ \left(-\mathbf{K}_{cc}^{(2)^{-1}}\mathbf{K}_{cb}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{f}_b^{(2)} + \mathbf{K}_{cc}^{(2)^{-1}}\mathbf{f}_c^{(2)}\right) - \mathbf{K}_{aa}^{(2)^{-1}}\mathbf{K}_{ab}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{f}_b^{(2)} + \mathbf{K}_{aa}^{(2)^{-1}}\mathbf{f}_a^{(2)},$$

$$\hat{\mathbf{f}}_b^{(2)} = -\mathbf{K}_{ba}^{(2)}\hat{\mathbf{K}}_a^{(2)^{-1}}\hat{\mathbf{f}}_a^{(2)} - \mathbf{K}_{bc}^{(2)}\hat{\mathbf{K}}_c^{(2)^{-1}}\hat{\mathbf{K}}_{a2}^{(2)}\hat{\mathbf{K}}_a^{(2)^{-1}}\hat{\mathbf{f}}_a^{(2)} - \mathbf{K}_{bc}^{(2)}\hat{\mathbf{K}}_c^{(2)^{-1}}\hat{\mathbf{f}}_c^{(2)} + \mathbf{f}_b^{(2)},$$

$$\hat{\mathbf{K}}_c^{(2)} = \mathbf{I} - \mathbf{K}_{cc}^{(2)^{-1}}\mathbf{K}_{cb}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{K}_{bc}^{(2)},$$

$$\hat{\mathbf{K}}_{a2}^{(2)} = -\mathbf{K}_{cc}^{(2)^{-1}}\left[\mathbf{K}_{ca}^{(2)} - \mathbf{K}_{cb}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{K}_{ba}^{(2)}\right],$$

$$\hat{\mathbf{f}}_c^{(2)} = -\mathbf{K}_{cc}^{(2)^{-1}}\mathbf{K}_{cb}^{(2)}\mathbf{K}_{bb}^{(2)^{-1}}\mathbf{f}_b^{(2)} + \mathbf{K}_{cc}^{(2)^{-1}}\mathbf{f}_c^{(2)}.$$

The equilibrium equations at the boundary regions of the initial system and the first subsystem are obtained by substituting Eqns. (48), (50a) and (46a) into the equilibrium equations of Eqn. (15). And the displacements at the interior region of the initial system are calculated by solving Eqn. (47) from the displacements at its boundary. Repeating such process between two adjacent subsystems such as the first and second subsystems, the second and third subsystems, etc., we can obtain the equilibrium equations on the boundary regions of all subsystems. They can be written as

$$\begin{bmatrix} \hat{\mathbf{K}}^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & \hat{\mathbf{K}}_a^{(2)} & 0 & \cdots & 0 \\ 0 & 0 & \hat{\mathbf{K}}_c^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \hat{\mathbf{K}}^{(s)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_b^{(1)} \\ \mathbf{u}_a^{(2)} \\ \mathbf{u}_c^{(2)} \\ \vdots \\ \mathbf{u}_a^{(s)} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{f}}^{(1)} \\ \hat{\mathbf{f}}_a^{(2)} \\ \hat{\mathbf{f}}_c^{(2)} + \hat{\mathbf{K}}_{a2}^{(2)}\hat{\mathbf{f}}_c^{(2)} \\ \vdots \\ \hat{\mathbf{f}}^{(s)} \end{bmatrix}, \tag{51}$$

where $\hat{\mathbf{K}}^{(i)} = \mathbf{I} - \mathbf{K}_{bb}^{(i)^{-1}}\mathbf{K}_{ba}^{(i)}\mathbf{K}_{aa}^{(i)^{-1}}\mathbf{K}_{ab}^{(i)}$, $\hat{\mathbf{K}}^{(s)} = \mathbf{I} - \mathbf{K}_{aa}^{(s)^{-1}}\mathbf{K}_{ab}^{(s)}\mathbf{K}_{bb}^{(s)^{-1}}\mathbf{K}_{ba}^{(s)}$,

$$\hat{\mathbf{f}}^{(i)} = \mathbf{K}_{bb}^{(i)-1}\mathbf{f}_b^{(i)} - \mathbf{K}_{bb}^{(i)-1}\mathbf{K}_{ba}^{(i)}\mathbf{K}_{aa}^{(i)-1}\mathbf{f}_a^{(i)},\ \hat{\mathbf{f}}_{(s)} = \mathbf{K}_{aa}^{(s)-1}\mathbf{K}_{ab}^{(s)}\mathbf{K}_{bb}^{(s)-1}\mathbf{f}_b^{(s)} - \mathbf{K}_{aa}^{(s)-1}\mathbf{f}_a^{(s)},$$

$$\hat{\mathbf{K}}_{a2}^{(i)} = -\mathbf{K}_{cc}^{(i)-1}\left[\mathbf{K}_{ca}^{(i)} - \mathbf{K}_{cb}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{K}_{ba}^{(i)}\right],\ \hat{\mathbf{K}}_c^{(i)} = \mathbf{I} - \mathbf{K}_{cc}^{(i)-1}\mathbf{K}_{cb}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{K}_{bc}^{(i)},$$

$$\hat{\mathbf{f}}_c^{(i)} = -\mathbf{K}_{cc}^{(i)-1}\mathbf{K}_{cb}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{f}_b^{(i)} + \mathbf{K}_{cc}^{(i)-1}\mathbf{f}_c^{(i)},$$

$$\hat{\mathbf{f}}_a^{(i)} = -\mathbf{K}_{aa}^{(i)-1}\left(\mathbf{K}_{ab}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{K}_{bc}^{(i)} - \mathbf{K}_{ac}^{(i)}\right)\left(\mathbf{I} - \mathbf{K}_{cc}^{(i)-1}\mathbf{K}_{cb}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{K}_{bc}^{(i)}\right)^{-1}\left(-\mathbf{K}_{cc}^{(i)-1}\mathbf{K}_{cb}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{f}_b^{(i)} + \mathbf{K}_{cc}^{(i)-1}\mathbf{f}_c^{(i)}\right)$$
$$-\mathbf{K}_{aa}^{(i)-1}\mathbf{K}_{ab}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{f}_b^{(i)} + \mathbf{K}_{aa}^{(i)-1}\mathbf{f}_a^{(i)},$$

$$\hat{\mathbf{K}}_a^{(i)} = \mathbf{K}_{aa}^{(i)-1}\left(\mathbf{K}_{ab}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{K}_{bc}^{(i)} - \mathbf{K}_{ac}^{(i)}\right)\left(\mathbf{I} - \mathbf{K}_{cc}^{(i)-1}\mathbf{K}_{cb}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{K}_{bc}^{(i)}\right)^{-1}\left[\mathbf{K}_{cc}^{(i)-1}\left(\mathbf{K}_{ca}^{(i)} - \mathbf{K}_{cb}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{K}_{ba}^{(2i)}\right)\right]$$
$$+\mathbf{I} - \mathbf{K}_{aa}^{(i)-1}\mathbf{K}_{ab}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{K}_{ba}^{(i)}, \qquad i = 1,\quad 2,\quad \cdots,\quad s$$

where $(i)$ indicates the i-th subsystem.

The static behavior at the constrained boundary regions on the s subsystems is described by substituting the equilibrium equations of Eqn. (51) and the compatibility of Eqns. (46) into Eqn. (15). The equilibrium equations to describe the interior regions of the s subsystems can be assembled as

$$\begin{bmatrix} \mathbf{K}_{aa}^{(1)} & 0 & \cdots & 0 \\ 0 & \mathbf{K}_{bb}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{K}_{bb}^{(s)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_a^{(1)} \\ \mathbf{u}_b^{(2)} \\ \vdots \\ \mathbf{u}_b^{(s)} \end{bmatrix} = \begin{bmatrix} -\left(\mathbf{K}_{ab}^{(1)}\mathbf{u}_b^{(1)} - \mathbf{f}_a^{((1))}\right) \\ \hat{\mathbf{f}}_b^{(2)} \\ \vdots \\ \hat{\mathbf{f}}_b^{(s)} \end{bmatrix} \tag{52}$$

where $\hat{\mathbf{f}}_b^{(i)} = -\mathbf{K}_{ba}^{(i)}\hat{\mathbf{K}}_a^{(i)-1}\hat{\mathbf{f}}_a^{(i)} - \mathbf{K}_{bc}^{(i)}\hat{\mathbf{K}}_c^{(i)-1}\hat{\mathbf{K}}_{a2}^{(i)}\hat{\mathbf{K}}_a^{(i)-1}\hat{\mathbf{f}}_a^{(i)} - \mathbf{K}_{bc}^{(i)}\hat{\mathbf{K}}_c^{(i)-1}\hat{\mathbf{f}}_c^{(i)} + \mathbf{f}_b^{(i)}$,

$\hat{\mathbf{K}}_{a2}^{(i)} = -\mathbf{K}_{cc}^{(i)-1}\left[\mathbf{K}_{ca}^{(i)} - \mathbf{K}_{cb}^{(i)}\mathbf{K}_{bb}^{(i)-1}\mathbf{K}_{ba}^{(i)}\right]$.

The derived solutions can be utilized based on the assumption that the stiffness matrices are full-rank. The following section considers the method to synthesize the floating subsystems of free-free end conditions.

### 3.2 Static synthesis of floating substructures

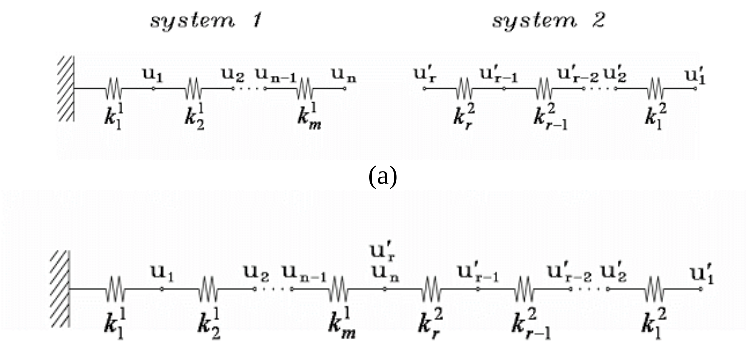
The derived method was obtained based on the assumption that the stiffness matrix is positive-definite full-rank matrix. When the substructure has rigid body modes, the stiffness matrix becomes semi-positive definite. The method can not handle the synthesis of unstable and stable structures because the unstable structure has the stiffness matrix of rank deficiency. The derived method should be modified to utilize in describing the static responses to synthesize partitioned floating substructures.

Consider the synthesis of a stable fixed-free system 1 and an unstable free-free system 2 to be depicted in Fig. 2(a). The systems have a single DOF at each node, horizontal displacement. The two systems are interconnected at nodes $n$ or $r$. The equilibrium equations of the systems 1 and 2 are written as

$$\begin{bmatrix} k_1^1+k_2^1 & -k_2^1 & \cdots & 0 & 0 \\ -k_2^1 & k_2^1+k_3^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & k_{n-1}^1+k_n^1 & -k_n^1 \\ 0 & 0 & 0 & -k_n^1 & k_n^1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ 0 \end{bmatrix}, \tag{53a}$$

$$\begin{bmatrix} k_1^2 & -k_2^2 & \cdots & 0 & 0 \\ -k_2^2 & k_2^2+k_3^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & k_{r-1}^2+k_r^2 & -k_r^2 \\ 0 & 0 & 0 & -k_r^2 & k_r^2 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_{r-1}' \\ u_r' \end{bmatrix} = \begin{bmatrix} f_1' \\ f_2' \\ \vdots \\ f_{r-1}' \\ f_r' \end{bmatrix}, \tag{53b}$$

where $u_i$ $(i=1,\ 2,\ \cdots,\ n)$ and $u_j'$ $(j=1,\ 2,\ \cdots,\ r)$ are the displacements of the systems 1 and 2, respectively. And $f_i$ and $f_j'$ represent the applied forces of the systems 1 and 2, respectively.




(a)

(b)

**Fig. 2.** A synthesis of fixed-free end and free-free end substructures; (a) two substructures, (b) a synthesized entire structure

The subsystem 2 has free-free end conditions and the rank of its stiffness is $r-1$. The two systems can not be synthesized based on the derived equation because the stiffness matrix is not full-rank, and the equation should be modified. The entire system of $(n+r)$ DOFs of the systems 1 and 2 is described by the $(n+r-1)$ displacements excluding the common displacement at the interface. The compatibility condition at the interface between adjacent subsystems can be written as

$$u_n = u_r'. \tag{54}$$

As shown in Eqn. (54), the displacement at the interface is described by a single displacement $u_n$ or $u_r'$. Extracting only the equilibrium equations at the interface from the last equations of Eqns. (53a) and (53b), they can be written as

$$\begin{bmatrix} k_n^1 & 0 \\ 0 & k_r^2 \end{bmatrix}\begin{bmatrix} u_n \\ u_r^{'} \end{bmatrix} = \begin{bmatrix} k_n^1 u_{n-1} \\ f_r^{'} + k_r^2 u_{r-1}^{'} \end{bmatrix}. \tag{55}$$

The modified stiffness matrix exhibits a full rank, and the substitution of Eqns. (54) and (55) into Eqn. (15) yields

$$\begin{bmatrix} u_n \\ u_r^{'} \end{bmatrix} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} k_n^1 & 0 \\ 0 & k_r^2 \end{bmatrix}^{-1/2} \left( \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} k_n^1 & 0 \\ 0 & k_r^2 \end{bmatrix}^{-1/2} \right)^{+} \begin{bmatrix} 1 & -1 \end{bmatrix} \right) \begin{bmatrix} u_{n-1} \\ f_r^{'} + u_{r-1}^{'} \end{bmatrix}$$

$$= \frac{1}{k_n^1 + k_r^2} \begin{bmatrix} k_n^1 & k_r^2 \\ k_n^1 & k_r^2 \end{bmatrix} \begin{bmatrix} u_{n-1} \\ f_r^{'} + u_{r-1}^{'} \end{bmatrix}. \tag{56}$$

Thus, the two displacements can be calculated as

$$u_n = u_r^{'} = \frac{1}{k_n^1 + k_r^2} \left[ k_n^1 u_{n-1} + k_r^2 \left( f_r^{'} + u_{r-1}^{'} \right) \right]. \tag{57}$$

The entire system is explicitly described by total $(n+r-1)$ equations of Eqn. (57) and $(n+r-2)$ equations to insert Eqn. (57) into Eqns. (53). From the derivation, it is known that the proposed method can be utilized in the synthesis method of substructures to include floating substructures.

The following considers the synthesis of substructures bonded at many overlapped interfaces. For two substructures with overlap as depicted in Fig. 3, neighboring substructures are allowed to have common, or overlap, members. The system has a DOF at each node, horizontal displacement. An end must be supported to be a stable structure. Let us assume that the two substructures 1 and 2 have $m(m < n, m < r)$ common points of the substructures 1 and 2 to have $n$ and $r$ DOFs, respectively. The equilibrium equations are expressed as

$$\begin{bmatrix} \mathbf{K}_{ii}^1 & \mathbf{K}_{ib}^1 \\ \mathbf{K}_{bi}^1 & \mathbf{K}_{bb}^1 \end{bmatrix} \begin{bmatrix} \mathbf{u}_i^1 \\ \mathbf{u}_b^1 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_i^1 \\ \mathbf{0} \end{bmatrix}, \tag{58a}$$

$$\begin{bmatrix} \mathbf{K}_{ii}^2 & \mathbf{K}_{ib}^2 \\ \mathbf{K}_{bi}^2 & \mathbf{K}_{bb}^2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_i^2 \\ \mathbf{u}_b^2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_i^2 \\ \mathbf{f}_b^2 \end{bmatrix}, \tag{58b}$$

where $i$ and $b$ indicate internal and boundary DOFs, respectively. The substructure 2 is a floating structure with the stiffness matrix of rank $(r-1)$ and the entire structure is described in $(n+r-m)$ configuration space.

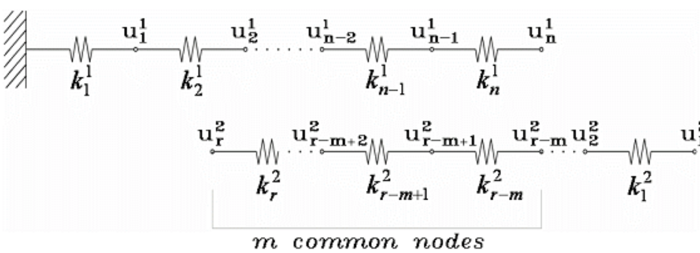



**Fig. 3.** Two overlapped substructures

The $m$ compatibility conditions between the neighboring substructures can be written as

$$\mathbf{u}_b^1 = \mathbf{u}_b^2, \tag{59}$$

where $\mathbf{u}_b^1$ and $\mathbf{u}_b^2$ are $m \times 1$ vectors. The $2m$ equilibrium equations related to the boundary displacements of the two substructures are written as

$$\begin{bmatrix} \mathbf{K}_{bb}^1 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{bb}^2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_b^1 \\ \mathbf{u}_b^2 \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{bi}^1 \mathbf{u}_i^1 \\ \mathbf{f}_b^2 - \mathbf{K}_{bi}^2 \mathbf{u}_i^2 \end{bmatrix}. \tag{60}$$

The compatibility conditions of Eqn. (59) can be written in a matrix form

$$\mathbf{A}\mathbf{u}_b = \mathbf{0}, \tag{61}$$

where A is an $m \times 2m$ Boolean matrix whose elements are 0, -1 or 1, and $\mathbf{u}_b = \begin{bmatrix} \mathbf{u}_b^{1^T} & \mathbf{u}_b^{2^T} \end{bmatrix}^T$. The substitution of Eqns. (60) and (61) into Eqn. (15) leads to the equilibrium equations expressed by the displacements at the interfaces to satisfy the compatibility conditions. Inserting their substitution into Eqns. (58) which are the equations to describe the interior regions, the static behavior of the entire structure can be obtained.

### 3.3 Dynamic synthesis of subsystems

By the similar process as the static approach, the dynamic equations of each substructure are firstly established and then they are combined based on the constraints of the dynamic responses at the interfaces between considered substructures. Assuming that the substructures in Fig. 1(a) are dynamic systems, let us consider the dynamic synthesis of the two dynamic substructures. The dynamic equations of two substructures can be written as

$$\begin{bmatrix} \mathbf{M}_{aa}^{(1)} & \mathbf{M}_{ab}^{(1)} \\ \mathbf{M}_{ba}^{(1)} & \mathbf{M}_{bb}^{(1)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_a^{(1)} \\ \ddot{\mathbf{u}}_b^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{aa}^{(1)} & \mathbf{K}_{ab}^{(1)} \\ \mathbf{K}_{ba}^{(1)} & \mathbf{K}_{bb}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_a^{(1)} \\ \mathbf{u}_b^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_a^{(1)} \\ \mathbf{p}_b^{(1)} \end{bmatrix}, \tag{62a}$$

$$\begin{bmatrix} \mathbf{M}_{aa}^{(2)} & \mathbf{M}_{ab}^{(2)} \\ \mathbf{M}_{ba}^{(2)} & \mathbf{M}_{bb}^{(2)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_a^{(2)} \\ \ddot{\mathbf{u}}_b^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{aa}^{(2)} & \mathbf{K}_{ab}^{(2)} \\ \mathbf{K}_{ba}^{(2)} & \mathbf{K}_{bb}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_a^{(2)} \\ \mathbf{u}_b^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_a^{(2)} \\ \mathbf{p}_b^{(2)} \end{bmatrix}, \tag{62b}$$

where $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ denote the $r \times 1$ and $s \times 1$ displacement vectors, respectively, and the second and first dynamic equations of Eqns. (62a) and (62b) represent the dynamic equations at boundary regions. The constraints that the dynamic responses at the $m(m < r, m < s)$ interfaces between adjacent subsystems are the same can be written as

$$\mathbf{u}_b^{(1)} = \mathbf{u}_a^{(2)}. \tag{63}$$

Expressing the coupled dynamic equations of Eqns. (62) by the modal coordinates and the mode shape matrix corresponding to the first $m_1$ eigenvalues, they can be written as

$$\overline{\mathbf{M}}^{(1)}\ddot{\mathbf{q}}^{(1)} + \overline{\mathbf{K}}^{(1)}\mathbf{q}^{(1)} = \overline{\mathbf{p}}^{(1)}, \tag{64a}$$

$$\overline{\mathbf{M}}^{(2)}\ddot{\mathbf{q}}^{(2)} + \overline{\mathbf{K}}^{(2)}\mathbf{q}^{(2)} = \overline{\mathbf{p}}^{(2)}, \tag{64b}$$

where $\begin{bmatrix} \mathbf{u}_a^{(i)^T} & \mathbf{u}_b^{(i)^T} \end{bmatrix}^T = \boldsymbol{\varphi}^{(i)}\mathbf{q}^{(i)}$, $i = 1,2$, $\boldsymbol{\varphi}^{(1)}$ and $\boldsymbol{\varphi}^{(2)}$ are $r \times m_1$ and $s \times m_1$ mode shape matrices, respectively, $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are $m_1 \times 1$ modal coordinate vectors. And



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$$\overline{\mathbf{M}}^{(i)} = \boldsymbol{\varphi}^{(i)^T}\mathbf{M}\boldsymbol{\varphi}^{(i)}, \tag{65a}$$

$$\overline{\mathbf{K}}^{(i)} = \boldsymbol{\varphi}^{(i)^T}\mathbf{K}\boldsymbol{\varphi}^{(i)}, \tag{65b}$$

$$\overline{\mathbf{p}}^{(i)} = \boldsymbol{\varphi}^{(i)^T}\mathbf{p}^{(i)}, \qquad i=1,\ 2. \tag{65c}$$

It is observed that the dynamic equations of $(r+s)$ DOFs are reduced to the decoupled dynamic equations of $2m_1$ DOFs. And the constraint equations can be written as

$$\boldsymbol{\varphi}^{(1)}_{m\times m_1}\mathbf{q}^{(1)} = \boldsymbol{\varphi}^{(2)}_{m\times m_1}\mathbf{q}^{(2)}. \tag{66}$$

Substituting Eqns. (64a), (64b) and (66) into Eqn. (34), it is found that the dynamic responses of the entire system can be described based on the modal coordinate vectors.

## 4. Applications

4.1 Application 1

The validity of the proposed method is illustrated through two simple applications. First, consider a three-spring system of fixed-free end conditions shown in Fig. 4. Let us assume that the initial structure is partitioned by three substructures to be composed of a stable structure and two floating substructures. Inversely, the entire structure is formed by interconnecting the three substructures. The equilibrium equations of the substructures are expressed by

$$k_1 u_1 = f_1, \tag{67a}$$

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1^{'} \\ u_2^{'} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{67b}$$

$$\begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_2 \\ f_3 \end{bmatrix}. \tag{67c}$$

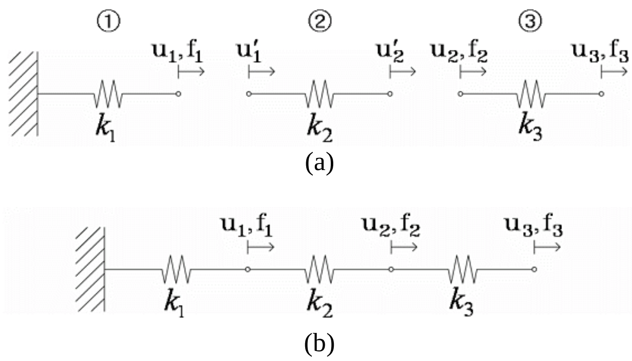


**Fig. 4.** A three-spring system; (a) three substructures, (b) an entire structure

Although the floating substructures are independently unstable, they can be stable by restricting the horizontal displacements or providing the forces for keeping the equilibrium state. For the synthesis of the substructures, the following compatibility conditions at nodes 2 and 3 are utilized:

$$u_1 = u_1^{'}, \tag{68a}$$

$$u_2 = u_2^{'}. \tag{68b}$$

Equations (68) are modified to the equilibrium equations with respect to the displacements at the interfaces of substructures:

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_1^{'} \end{bmatrix} = \begin{bmatrix} f_1 \\ k_2 u_2^{'} \end{bmatrix}, \tag{69a}$$

$$\begin{bmatrix} k_2 & 0 \\ 0 & k_3 \end{bmatrix} \begin{bmatrix} u_2^{'} \\ u_2 \end{bmatrix} = \begin{bmatrix} k_2 u_1^{'} \\ f_2 + k_3 u_3 \end{bmatrix}. \tag{69b}$$

Utilizing Eqns. (68) and (69) into Eqn. (15), and introducing $u_1 = u_1^{'}$ and $u_2 = u_2^{'}$ into the result, the final equations with the second equation of Eqn. (67c) are derived as

$$u_1 = \frac{1}{k_1 + k_2}\left( f_1 + k_2 u_2 \right), \tag{70a}$$

$$u_2 = \frac{1}{k_2 + k_3}\left( k_2 u_1 + k_3 u_3 + f_2 \right), \tag{70b}$$

$$u_3 = \frac{1}{k_3}\left( k_2 u_2 + f_3 \right)\cdot \tag{70c}$$

The derived results correspond with the equilibrium equations of the initial entire structure. Although the considered application is a simple structure, its concept can be easily extended to complicated structures with various interfaces.

## 4.2 Application 2

This application is to carry out the structural reanalysis of modified structure to add a bar to the initial truss structure. Consider a plane truss structure shown in Fig. 5. The nodal points and the members are numbered. Corresponding to each pair of nodal displacement components $(u_i, v_i)$ is expressed by a set of forces $(H_i, V_i)$. The initial truss structure is subjected to $100\text{kN}$ and $200\text{kN}$ in the downward and right-hand side direction at nodes. All members have elastic modulus of $200\text{GPa}$ and cross-sectional area of $2.5\times10^{-3}\text{m}^2$. The equilibrium equations of the truss are expressed by

$$10^8 \times \begin{bmatrix} 2.617 & 0 & 0 & -1.042 & 0 & -0.533 & -0.4 \\ 0 & 1.575 & 0.4 & 0 & 0 & -1.042 & 0 \\ 0 & 0.4 & 1.689 & 0 & 0 & 0 & 0 \\ -1.042 & 0 & 0 & 1.042 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 & 1.389 & 0 & -1.389 \\ -0.533 & -1.042 & 0 & 0 & 0 & 1.575 & 0.4 \\ -0.4 & 0 & 0 & 0 & -1.389 & 0.4 & 1.689 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{bmatrix} = 10^3 \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -100 \\ 200 \\ 0 \end{bmatrix}. \tag{71}$$

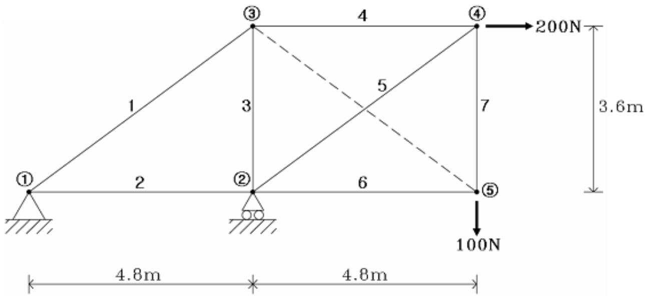


**Fig. 5.** A modified truss structure; The dotted line indicates the added bar

Premultiplying the inverse of stiffness matrix on both sides of Eqn. (71), the displacements of the initial truss can be obtained as

$$[u_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4 \quad u_5 \quad v_5]^T = [-1.3 \quad 7.6 \quad -1.8 \quad -1.3 \quad -20.2 \quad 10.8 \quad -19.4]^T \text{mm}. \tag{72}$$

Let us assume that we add a truss bar between nodes 3 and 5. The bar itself is unstable structure with free ends and can be stable structure by giving the forces for keeping the equilibrium state in plane. The forces are constraint forces and can be calculated by using compatibility conditions.

The equilibrium equations of the truss bar corresponding to the displacements $[u_3' \quad v_3' \quad u_4' \quad v_4']^T$ are written as

$$10^7 \times \begin{bmatrix} 5.3333 & -4.0 & -5.3333 & 4.0 \\ -4.0 & 3.0 & 4.0 & -3.0 \\ -5.3333 & 4.0 & 5.3333 & -4.0 \\ 4.0 & -3.0 & -4.0 & 3.0 \end{bmatrix} \begin{bmatrix} u_3' \\ v_3' \\ u_4' \\ v_4' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{73}$$

The stiffness matrix of Eqn. (73) exhibits the floating mode without supporting conditions.

Extracting the equilibrium equations corresponding to nodes 3 and 4 from Eqn. (71), respectively, they can be written as

$$10^8 \times \begin{bmatrix} 1.575 & 0.4 \\ 0.4 & 1.6889 \end{bmatrix} \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = 10^8 \times \begin{bmatrix} 1.0417u_5 \\ 0 \end{bmatrix} = 10^8 \times \begin{bmatrix} 0.013 \\ 0 \end{bmatrix}, \tag{74a}$$

$$10^8 \times \begin{bmatrix} 1.0417 & 0 \\ 0 & 1.3889 \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1.0417 \times 10^8 u_2 \\ 0.4 \times 10^8 u_2 - 100 \end{bmatrix} = 10^8 \times \begin{bmatrix} -0.0013 \\ 0.0135 \end{bmatrix}. \tag{74b}$$

And the equilibrium equations of the added bar of Eqn. (73) are modified by the stiffness matrix of full rank as

$$10^7 \times \begin{bmatrix} 5.333 & 0 \\ 0 & 3.0 \end{bmatrix} \begin{bmatrix} u_3' \\ v_3' \end{bmatrix} = -10^7 \times \begin{bmatrix} 0 & -4.0 & -5.33 & 4.0 \\ -4.0 & 0 & 4.0 & -3.0 \end{bmatrix} \begin{bmatrix} u_3' \\ v_3' \\ u_4' \\ v_4' \end{bmatrix}, \tag{75a}$$

$$10^7 \times \begin{bmatrix} 5.333 & 0 \\ 0 & 3.0 \end{bmatrix} \begin{bmatrix} u_4^{'} \\ v_4^{'} \end{bmatrix} = -10^7 \times \begin{bmatrix} -5.33 & 4.0 & 0 & -4.0 \\ 4.0 & -3.0 & -4.0 & 0 \end{bmatrix} \begin{bmatrix} u_3^{'} \\ v_3^{'} \\ u_4^{'} \\ v_4^{'} \end{bmatrix}. \tag{75b}$$

In order to establish the relationship between the initial truss and the bar, the compatibility conditions at nodes 3 and 4 between two structures are defined as

$$u_3 = u_3^{'},\ v_3 = v_3^{'}, \tag{76a}$$

$$u_4 = u_4^{'}, \quad v_4 = v_4^{'}. \tag{76b}$$

The substitution of Eqns. (74), (75) and (76) into Eqn. (15) yields the following:

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_3^{'} \\ v_3^{'} \end{bmatrix} = \begin{bmatrix} 0.4941u_5 + 0.253u_4^{'} - 0.1897v_4^{'} \\ -0.2011u_4^{'} + 0.1508v_4^{'} \end{bmatrix}, \tag{77a}$$

$$\begin{bmatrix} u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} u_4^{'} \\ v_4^{'} \end{bmatrix} = \begin{bmatrix} 0.7677u_2 + 0.2222v_5 + 0.2963u_3^{'} - 0.2222v_3^{'} - 0.00016 \\ 0.4187u_2 + 0.875v_5 - 0.1667u_3^{'} + 0.125v_3^{'} - 0.00063 \end{bmatrix}. \tag{77b}$$

Substituting Eqns. (77) into the displacements of the initial truss of Eqn. (71), the static displacements of the modified truss as well as the displacement variations due to the modification can be explicitly calculated. Starting from the displacements of initial structure, this application exhibits that the proposed method can determine the structural responses for changes in the design from the constrained displacements at the interfaces using compatibility conditions without solving the complete set of modified simultaneous equations.

## 5. Conclusions

The problem on the structural synthesis of substructures interconnected by interfaces and overlapped points is established by a mathematical system consisting of equilibrium equations and prescribed compatibility conditions. Assuming that the compatibility conditions are constraints to govern the static or dynamic responses between adjacent subsystems, this study derived the constrained static and dynamic equations to describe their responses. The approach is carried out by partitioning into two regions of interior and boundary regions, and giving the compatibility conditions. The approaches can be extended to the synthesis of the unstable subsystems of free-free end conditions and the validity of the proposed method was illustrated in several applications.

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