581. Mathematical formulation of instability of a subsea suspendent pipeline

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Abstract: Lateral vibration of underwater suspended pipeline was investigated for the case of pipeline oscillation due to vortex shedding. Firstly, tension force was defined at the connection legs on sea bottom. To define the dynamical equation the analogy of the Mathieu equation was applied, meanwhile Ince-Strutt diagram was used for its solution. As a numerical example we used the behavior of pipeline in a project between Turkey and North Cyprus in the East Mediterranean Sea. Good agreement was found between the theoretical results and experimental data of Danish Hydraulic Institute (DHI).

Keywords: Subsea pipeline**,** dynamic behavior, Ince-Strutt diagram, oscillation

Symbols

L_0 –Length of leg, m;
L – Lengthening of leg, m;
ΔL – difference of lengthening, m ;
Δx – maximum horizontal displacement of pipeline, m;
F_0 – unite tension force in the leg, kN;
F – Tension force in lengthening leg, kN;
φ – Angle of displacement, grad;
A_k – cross-section of leg, m^2;
F_x – horizontal projection of F – tension force, kN;
P – external wave force, kN;
μ – safety coefficient;
u_{adm} – permissible horizontal displacement;
F_{adm} – permissible tension force in the leg;
ω – Cyclic frequency of structure, rad/s;
θ – Frequency of the external force, rad/s;
T – Period of structure, s;
T_0 – period of the external force, s;
g – Gravitation acceleration, m/s^2;
ℓ – Length of the pipeline section, m;
E – Modulus of elasticity, kN/mm^2;
μ – Poisson ratio;
δ – Thickness of pipeline, m;
R – External radius of pipeline, m;
D – External diameter, m;
ρ – density of HDPE material, kg/m^3;
ρ_0 – density of water, kg/m^3;
M – mass of structure plus added water mass on the one meter, N/m;
I – moment of inertia of pipeline, m^4;
EI – stiffness of pipeline structure, kN.mm^2;
F_k – Karman force;
C_k – non-dimensional Karman coefficient (for cylinders $C_k \approx 1$);
S – area of the cross-section of pipeline;
ω_k – circular frequency of Karman vortex;
T_v – vortex shedding period;
Γ – vortex of strength magnitude;
U_∞ – incident velocity at the upstream end of the flow field;
A – amplitude of pipeline displacement
N – kinematical viscosity;
Re – Reynolds number.

Introduction:

In this research, we defined oscillation of suspended subsea pipelines [1] by analogy with suspended bridges and offshore tension leg platforms [2,4,5,9]. In this research the mathematical application of Mathieu equation with its numerical solution method are given [3]. The subsea vibration of long cylindrical body has many solutions in the technical literature [2,5,8]. But the dynamic equations for these structures have non-linear characteristics. Therefore, to solve the equations researchers must apply various numerical methods for investigating pipeline stability. The main problem is to solve the stability of a system that vibrates during vortex shedding [6,7]. The appropriate finite element code is given for comparing the accuracy of the obtained solution with the analytical one by different authors [15-17]. As an example we used a pipeline that extends between Turkey and North Cyprus [1] (Fig. 1).

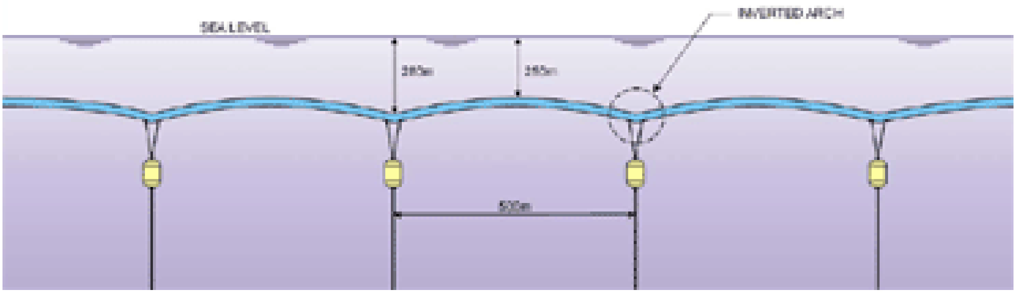

Fig. 1. Cyprus Peace water

Statement of the Problem

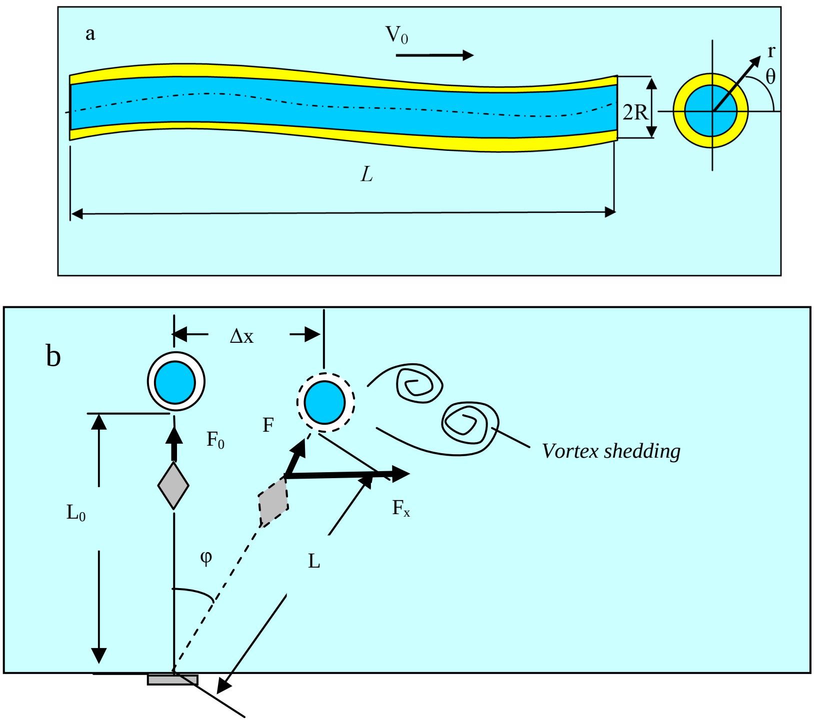

Fig. 2. The design scheme of the pipeline:
a) in longitudinal direction, b) cross-section of pipe showing the vortex shedding (where x – differential index by displacement, V_0 – mean velocity gradient from upstream to downstream direction at the outside of the pipeline, L_0 – length of the vertical connection length)

The dynamic equation of pipeline with damping is given as [2,4,7]:

$$Mu_{tt} + C|u_t|u_t - Fu_{xx} = 0 \tag{1}$$

where: M – mass of pipeline structure together with added water mass; F – tension force in leg; u – horizontal displacement; t and x – differential index by time and displacement; C –strength constant, $C = 0.5C_D \rho_w D$; C_D – hydrodynamic strength coefficient; ρ_w – water density; D – diameter of leg.

Fig. 2 a, b defines the loads that affect the dynamic stability of the pipeline during vortex shedding.

If the damping effect is neglected, we can substitute the Eq. (1):

$$Mu_{tt} - Fu_{xx} = 0 \tag{2}$$

and assume that

$$F = F_0 - F_1 \cos \omega t \tag{3}$$

Displacement of the pipeline can be written as $u = y(t)\sin\left(\frac{m\pi}{\ell}\right)$, where $y(t)$ – amplitude of harmonic displacement dependent on time; m – number of modes; L – length of pipe.
Then, if we substitute Eq. (2) into the Eq. (3) we may write

$$u_{tt} - \frac{F_0}{M}\left(1 - \frac{F_1}{F_0}\cos\theta t\right)u_{xx} = 0 \tag{4}$$

Different modes are illustrated in Fig. 3.
The result is:

$$y_{tt} + \left(\frac{m\pi}{\ell}\right)^2\left[\frac{F_0}{M}\left(1 - \frac{F_1}{F_0}\cos\theta t\right)\right]y = 0 \tag{5}$$

If we put in Eq. (5) instead of the θt value, we find another parameter $\tau = \frac{\omega t}{2}$ giving $\cos\theta t$ as $\cos 2\tau$, where θ – frequency of the external force,

$$y_{tt} + \left(4m^2\frac{\omega^2}{\theta^2} - 4m^2\frac{\omega^2}{\theta^2}\frac{F}{F_0}\cos 2\tau\right)y = 0 \tag{6}$$

where $\omega = \left(\frac{\pi}{\ell}\right)\sqrt{\frac{F_0}{M}}$ is circular frequency of lateral vibration of the pipeline system.

Equation (6) is known as Mathieu equation. In canonical form we can write it as follows [3, 6]:

$$y_{tt} + (a - 2q\cos 2\tau)y = 0 \tag{7}$$

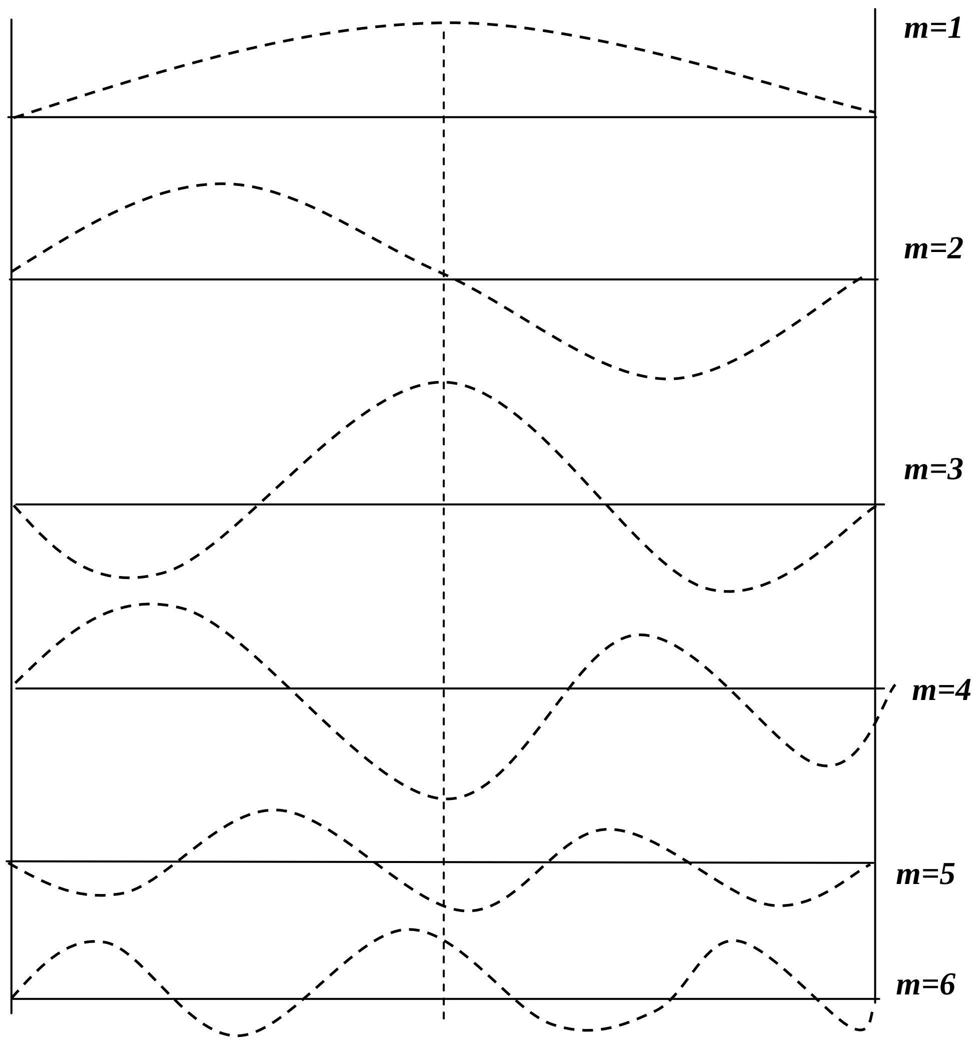

Fig. 3. Different modes of the dynamical stability of the pipeline

where *a* and *q* are constants. From Eq. (6) we can write [3,6]

$$a = 4m^2 \frac{\omega^2}{\theta^2}; \qquad q = 2m^2 \frac{\omega^2}{\theta^2}\frac{F}{F_0} = \frac{a}{2}\frac{F}{F_0} \tag{8}$$

If the force changes with harmonic law $P = P_0 + P_t\Phi(t)$, where P – external wave force; P_0 – unit wave force; P_t – wave force that is independent of time; Φ – function of time; T – period of wave motion. $\Phi(t+T) = \Phi(t)$. Then this equation can be given: the Hill Equation [13, 14]:

$$y_{tt} + \omega^2 \left[1 - 2q\Phi(t)\right] y = 0 \tag{9}$$

The Mathieu equation has an oscillating nature, and depends on constants *a* and *q*: two solutions have stable and instable character (Fig. 4).

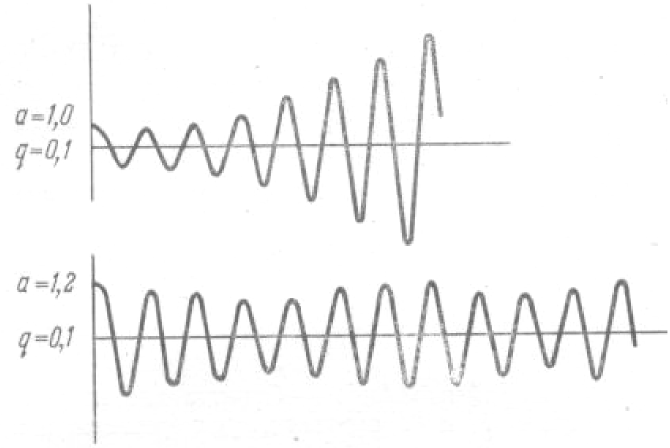

Fig. 4. Two solution of Mathieu equation: a) instable; b) stable [4]

The domains of stability for the solution of the Mathieu equation are given in the Ince-Strutt diagram (Fig. 5). The solution of the Mathieu equation to contact with the subsea pipeline instability is given below in the Eqs. (21-27), which is solved by diagram (Fig. 5) and by theoretical background .

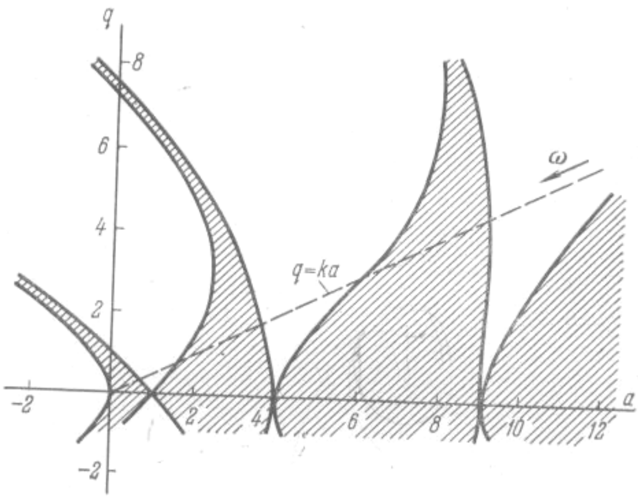

Fig. 5. Ince-Strutt diagram [6]

Every curve of the graph is given by the Mathieu function. At first among four instable fields we can write exact equations, if we mark them as a_n^r and a_n^l (in this *r* index is right, and the *l* index is left hand side) as [13, 14]:

$$\left.\begin{aligned}
a_0^r &= -\frac{1}{2}q^2 + \frac{7}{128}q^4 - \dots,\\
a_1^r &= 1 + q - \frac{1}{8}q^2 - \frac{1}{64}q^3 - \frac{1}{1536}q^4 + \dots,\\
a_1^l &= 1 - q - \frac{1}{8}q^2 + \frac{1}{64}q^3 - \frac{1}{1536}q^4 - \dots,\\
a_2^r &= 4 + \frac{5}{12}q^2 - \frac{763}{13824}q^4 + \dots,\\
a_2^l &= 4 - \frac{1}{12}q^2 + \frac{5}{13824}q^4 - \dots,\\
a_2^r &= 9 + \frac{1}{16}q^2 + \frac{1}{64}q^3 + \frac{13}{20480}q^4 + \dots,\\
a_3^l &= 9 + \frac{1}{16}q^2 - \frac{1}{64}q^3 + \frac{13}{20480}q^4 - \dots
\end{aligned}\right\} \tag{10}$$

In the shaded area the stable domains are given. In the shaded areas of the Ince-Strutt diagram we have parametric vibration case for different position of the pipeline. From Eq. (8) we observe that frequency of the system θ is larger as if a and q are smaller. So, the relationship of these parameters has constant values by the state of systems as $q = ka$ may be defined from points on the diagram as a line (Fig. 5).

Vortex shedding

In the starting process of separated flow around a circular cylinder a symmetric wake domain develops, but due to instabilities, asymmetry will soon occur. The consequence is that vortices are alternatively shed from each side of the cylinder depending on the cross-section of the pipeline [11]. Under shock wave forces and, as a consequence, Karman vortex shedding from the pipeline has a horizontal displacement like the Δx. Then the legs of structure have tension effect on the ΔL value (Fig. 2). The frequency effect of vortex shedding is defined by the formula $\theta = 0.22\frac{V}{D}$, where V – velocity of wind wave; D – diameter of pipeline. The coefficient 0.22 is the Strouhal number for a circular section of the pipeline [4,11]. The force affecting the Karman vortex for rigid cylinders is:

$$F_k = C_k\left(\frac{1}{2}\rho_0 v^2 S\right)\sin\omega_k t = F_{0k}\sin\omega t \tag{11}$$

where F_k – Karman force; C_k – non-dimensional Karman coefficient (for cylinders $C_k \approx 1$); S – area of the cross-section of pipe line; ρ_0 – density of water; ω_k – circular frequency of the Karman vortex. Considering a long circular cylinder, the frequency of vortex shedding is given by the empirical formula [11]:

$$\frac{\theta d}{V} = 0.198\left(1 - \frac{19.7}{\mathrm{Re}}\right) \tag{12}$$

where θ is vortex shedding frequency, Re is Reynolds number, $\mathrm{Re} = \frac{Vd}{\nu}$. This formula can be written generally between the range $250 < \mathrm{Re} < 2\times10^5$ which is in the transition region. Each vortex eddy is mathematically represented as a local vortex shedding of strength magnitude (Fig. 6).

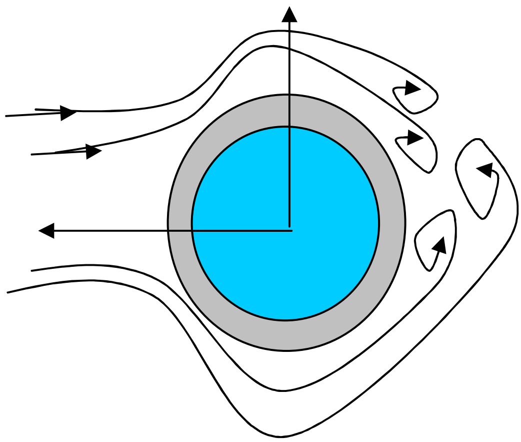

Fig. 6. Example of vortex shedding around the pipeline

Eddies in one row are either placed exactly on the opposite side from those of the other row or they are symmetrically staggered (Fig. 7). So, if the pipeline has long horizontal dimensions, the vortex shedding are arranged in zigzag patterns. The mathematical description of these lines is given in the complex form as [11, 12].

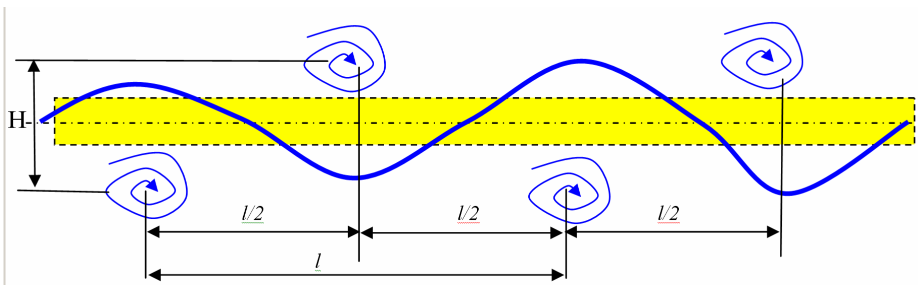

Fig. 7. Arrangement of vortices in a Von Karman vortex street

A stability investigation leads to the result that the first observation is given as instability of the system because of the vortex shedding around the boundary layer of the pipeline. The second observation has generally the unstable character, but becomes stable character for a definite ratio between the vortex street width h and distance l between two adjacent vortices in the same row:

$$\frac{h}{l} = \frac{1}{\pi}\cosh^{-1}\sqrt{2} = 0.28 \qquad (13)$$

From Fig. 7 we determine

$$T_v\left(U_\infty - \frac{\Gamma}{l\sqrt{8}}\right) = l \qquad (14)$$

where T_v – vortex shedding period; Γ – vortex of strength magnitude; U_∞ – incident velocity at the upstream end of the flow field. For simplicity let us put $h \approx D$, where D – the cylinder diameter and let us approximate the vortex velocities to U_∞ . Then we may write [11]:

$$T_v U_\propto = \frac{D}{0.28} \quad (15)$$

Thus, if the length between vortex-shedding *l* is much larger than *12R* (*R* – radius of pipe), then the flow field will be unstable. Experimental values of the mean relative spacing *h/l* vary between 0.19 and 0.3.

Problem solution

There are many solutions to the Mathieu equation: *Whitaker, Watson (1963) → a=b, q=-8c; Stratton (1942) →a=b-c²/2, 4q=c²; Yanke-Emde-Leush (1964) → a=4b, q=8c; National Bureau of Standard (1951) →a=b-c/2, q=c/4* [3,6,7,10,13,14].

From Fig. 2, if we have fixed support and no displacement of this point then the system is unstable. If the foundation has small motion then this system may be stabile. If we change the sign of the Eq. (6) then accordingly to Eq. (8) we can write:

$$a = -4m^2 \frac{\omega^2}{\theta^2} \quad (16)$$

From the diagram (Fig. 8) we can see that *a* parameter depends on vibration amplitude. Then amplitude *A* has a small mass (pipeline) which will be unstable, that is $a=m^2$ or *a=1,4,9,…*. which is given as $\omega_1 = 2\sqrt{\frac{g}{l}};\omega_2 = \sqrt{\frac{g}{l}};\omega_3 = \frac{2}{3}\sqrt{\frac{g}{l}};...$ for every number of modes.

The unstable field defined by *m=1* has a main field and much avoidable field because of the biggest displacement and has a practical value because the biggest oscillation mode. For definition instability of oscillation of system can be used for analogy for dependence of tension leg [2,5].

If we analyze Eq. (1) after different transformation we can define the amplitude of oscillation as:

$$a* = \frac{9\pi^2 M}{32C}\left[\frac{\omega^4 F^2}{\theta^4 F_0^2} - 4\left(\frac{\omega^2}{\theta^2} - \frac{1}{4}\right)^2\right]^{\frac{1}{2}} \quad (17)$$

For $\frac{\omega^2}{\theta^2} = \frac{1}{4}; or\left(\frac{T_0}{T} = \frac{1}{2}\right)$, where T_0 – period of pipeline structure; *T* – period of wave motion, relation of the maximum amplitude of displacement by lateral oscillation is:

$$a* = \frac{9\pi^2 MF}{128CF_0} \quad (18)$$

Its formula enables us to define maximum amplitude of pipeline oscillation from tension dependence during vibration – *F* to initial tension force $–F_0$.

During small amplitude, when $0 < |q| < 1$, the stability of pipeline may exist if it fulfills the condition $|a| < \frac{q^2}{2}$ (Eq.(8)).

In the non-linear systems the resonance appears from the following condition

$$\theta \approx \frac{p}{q}\omega \quad (19)$$

where *p* and *q* – whole prime numbers.

1) If $p = q = 1$, $\theta \approx \omega$ this case is a basic case or ordinary resonance.
2) If $q = 1$, $\theta \approx p\omega$ or $\omega = \frac{\theta}{p}$ - Parametric resonance. This resonance type may be given in the linear systems with periodic coefficients, too.
3) If $p = 1$, $\omega \approx qv$ - Resonance on the overtones for external frequency.

Eq. (7) is the basic de-multiplication resonance, where *p=1, q=2*. Then we have $\omega = \frac{\theta}{2}$.

In the first approximation we write

$$y = b\cos\left(\frac{\theta}{2}t + \psi\right) \tag{20}$$

where *b* and *θ* is defined from system of equations:

$$\left.\begin{aligned} \frac{db}{dt} &= -\frac{bq\omega^2}{\theta}\sin 2\psi \\ \frac{d\psi}{dt} &= \omega - \frac{\theta}{2} - \frac{q\omega^2}{\theta}\cos 2\psi \end{aligned}\right\} \tag{21}$$

If we introduce new change parameters *u* and *v*, then

$$u = b\cos\psi\,; \qquad v = b\sin\psi \tag{22}$$

The differential form of Eq.(19) we take into consideration the Eq.(20), we have

$$\left.\begin{aligned} \frac{du}{dt} &= \frac{db}{dt}\cos\psi - \frac{d\psi}{dt}b\sin\psi = \left[-\frac{q\omega^2}{\theta} - \left(\omega - \frac{\theta}{2}\right)\right]b\sin\psi \\ \frac{dv}{dt} &= \frac{db}{dt}\sin\psi + \frac{d\psi}{dt}b\cos\psi = \left[-\frac{q\omega^2}{\theta} + \left(\omega - \frac{\theta}{2}\right)\right]b\cos\psi \end{aligned}\right\} \tag{23}$$

or

$$\left.\begin{aligned} \frac{du}{dt} &= \left[-\frac{q\omega^2}{\theta} - \left(\omega - \frac{\theta}{2}\right)\right]b\sin\psi \\ \frac{dv}{dt} &= \left[-\frac{q\omega^2}{\theta} + \left(\omega - \frac{\theta}{2}\right)\right]b\cos\psi \end{aligned}\right\} \tag{24}$$

The solution of Eq. (23) with substitution of Eq. (19) the equation system is dependent on the roots of the characteristic equation

$$\begin{vmatrix} \lambda & \frac{q\omega^2}{2\theta} + \left(\omega - \frac{\theta}{2}\right) \\ \frac{q\omega^2}{2\theta} - \left(\omega - \frac{\theta}{2}\right) & \lambda \end{vmatrix} = 0 \tag{25}$$

or

$$\lambda^2 - \frac{q^2\omega^4}{4\theta^2} + \left(\omega - \frac{\theta}{2}\right)^2 = 0 \tag{26}$$

Then the mean square of the equation gives:

$$\lambda = \sqrt{\frac{q^2\omega^4}{4\theta^2} - \left(\omega - \frac{\theta}{2}\right)^2} \tag{27}$$

Thus, if the frequency of external force in the following interval is

$$2\omega\left(1 - \frac{q}{4}\right) < \theta < 2\omega\left(1 + \frac{q}{4}\right) \tag{28}$$

then this system may give rise to parametric resonance and the amplitude of vibration will increase exponentially. This equality has an unstable field. Now we define the amplitude b and vibration rotation ψ.

$$\left.\begin{aligned} b^2 &= u^2 + v^2 \\ \theta &= \arctan\frac{v}{u} \end{aligned}\right\} \tag{29}$$

According to Eqs. (22) and (24) we can see that by imaginary λ the amplitude b will be limited by a time function. If λ, the amplitude b will to increase by an exponential law. If in this system $y=0$ the state is unstable and the system can self-oscillate.

Selected (used) data

As a representative example for solution of the aforementioned problem we used the pipeline between Turkey and North Cyprus located at the narrowest part of the strait formed by Turkey and North Cyprus. The pipeline will provide water at a rate of 75 million m³ per year (2.38 m³/s). The pipeline will be a submerged floating structure and the subsea section of the pipeline will consist of 1.6 m diameter HDPE (High Density Polyethylene) pipe approximately 78 km long. In the shore approach sections of the route, the pipeline will be either resting on the seabed or be trenched and backfilled below seabed level. Between the 250 m depth contours on both the Turkish and Cyprus sides, the pipeline will be suspended at a water depth of at least 250 m. The pipeline will span from vertical legs anchored to the sea bed in spans of approximately 400 – 500 meters length.

Numerical Results

We performed a numerical simulation using the following real data taken from the project [1]. The length of pipe for one section $l=500$ *m;* radius $R=0.8$ *5m* $(D=1.7$ *m).* The thickness of pipe $\delta=0.063$ *m*. The Poisson ratio is $\nu=0.44$. The density of HDPE material of pipe $\rho=1.4x10^3$ *kg/m*^{3}. The density of sea water $\rho_0=1.03x10^3$ *kg/m*^{3}. The elasticity modulus of material $E=120000$ *t/m*^{2}. The stiffness of pipe $EI=7500$ *kN·mm*^{2}. The initial tension of legs was as $F_0=600;\ 800;\ 1000$ *kN.* The mass of pipe on the unit is $M=600$ *N/m.* If the point $(a;\ q)$ in the shaded domain of the stability graph is found then the Mathieu equation has the following relation (Fig. 5):

$$y = Ae^{i\sigma x}p_1(x) + Be^{-i\sigma x}p_2(x) \tag{30}$$

where A and B are integration constants; $p_1(x)$ and $p_2(x)$ are periodic functions with 2π period; σ – real value of outside modes of boundary layer, which is equally half of real value of the inside mode.

The main results of this calculation are provided in Table 1. The relationship between amplitude of displacement and frequency are presented in Fig. 8 and Fig. 9.

Table 1. The main results of calculation of Mathieu equation coefficients

F_0, kN	***ω/θ***	***m***	***a***	***q***	***State***
600	0.5	1	1	0.5	*Unstable*
		2	4	2	*unstable*
		3	9	3	*unstable*
800	0.57	1	1.3	0.65	*unstable*
		2	5.2	2.6	*unstable*
		3	11.7	5.85	*unstable*
1000	0.65	1	1.7	0.85	*unstable*
		2	6.76	3.38	*unstable*
		3	15.2	7.6	*unstable*

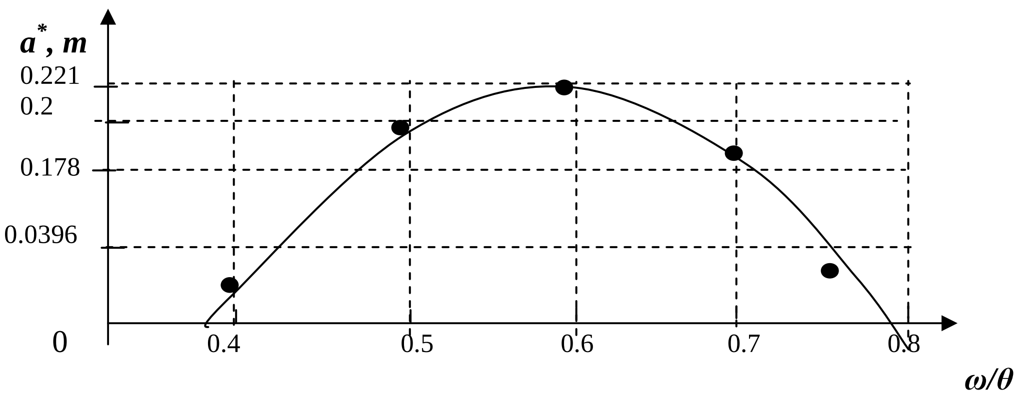

Fig. 8. The amplitude of parametric reaction of pipeline

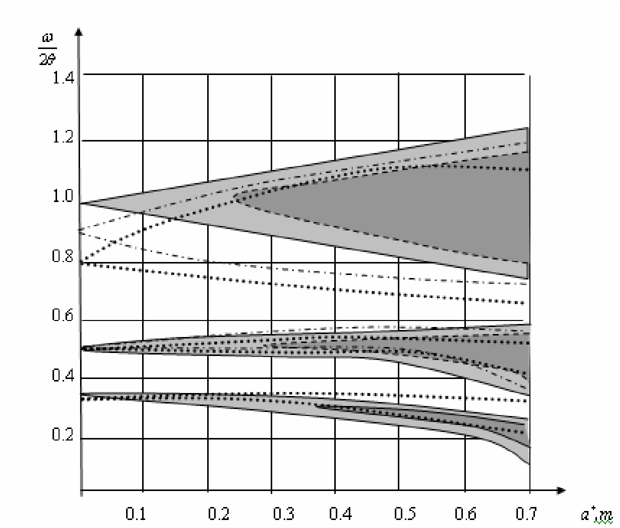

Fig. 9. The diagram of parametric resonance of pipeline: ▬▬ calculated; ▬ ▬ with damping $\frac{\varepsilon}{\omega} = 0.01$, dark grey [19] ; _._._. Papaidoussis & Issid, 1974 [18]; •••••• Intec Engineering Group-Danish Technology Institute, 2007 [1].

Discussion

During mathematical formulation of instability problem at undersea we investigated the stability problem of undersea pipeline and application of the Mathieu equation allowed determination of theoretical formulation of pipe instability because of vortex shedding. To solve the problem, first of all the vortex shedding effect on the pipe was provided. The Mathieu equation was solved theoretically by using the analogy with suspended bridges [4, 9], TLP-type platforms [2, 5] as well as floating offshore platforms [11] and applying the most common numerical methods [6,7,13,15,16,17]. The theoretical findings show good agreements with the actual measurement results.

Conclusion

- By analogy with suspended bridges it may be stated that the suspended undersea pipelines will experience vibration with frequency equal to half of the frequency of the wind wave load;
- During horizontal vortex shedding the pipeline looses dynamical stability and exhibits unstable character. Therefore it is necessary to calculate dynamical stability for such structures;
- This problem is a non-stationary and therefore the stability problem may be an example which cannot be analyzed by statics methods;
- Different modes of the dynamical stability of the pipeline are presented as symmetrical vibration modes $m=1, 3, 5, \ldots$, which indicate parametric resonance case from Eqs. (16)-(17). That can be observed only by $m=1, 3, 5, \ldots$.
- Application of Ince-Strutt diagram enables definition of coefficients a and q without solution of the Mathieu equation and can be defined by Mathieu functions with analytical methods;
- Numerical solutions indicate that all of the cases with different forces and modes are in an unstable state;
- In order to avoid these unstable cases appropriate engineering measures must be considered.

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