584. Modeling of dynamic stability of flexible ultrasonic waveguides

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Abstract. The article considers mathematical model intended for study of dynamic stability and parametric resonance of flexible ultrasonic waveguides for applications in technology and medicine. Considered problem is reduced to the well-known Mathieu equation applied in the theory of dynamic stability of elastic systems. Parameters of the Mathieu equation defining stability of the waveguide are determined by means of finite elements method using ANSYS software and APDL programming language.

Keywords: flexible waveguide, longitudinal vibrations, flexural vibrations

1. Introduction

This article is devoted to study of dynamic stability of flexible two-step waveguides for transmission of ultrasonic vibrations and should be considered as continuation of the cycle of authors' articles related to this problem [1, 2]. Loss of stability is a negative phenomenon during operation of waveguide systems with large slenderness ratio and it is manifested in the form of parametric resonance, i.e. generation of transverse vibrations with considerable amplitude induced by longitudinal vibrations and having circular frequency $\omega = n\Omega/2$, where Ω is circular frequency of longitudinal vibrations, n is a natural number [3, 4]. Parametric resonance can lead to the fracture of the waveguide system and also causes decrease in efficiency of its operation in the cases when intensifying action of ultrasound is determined by the amplitude of longitudinal vibrations. In this relation it is of interest to study effect of parameters of the waveguide system on its dynamic stability.

2. Problem formulation

The object of this study is a waveguide consisting of two cylindrical sections (steps) with a constant cross-section connected by the smooth transitional section of Fourier horn type. Profile of the transitional section is found by empirical way and described by polynomial function [1, 2]. Step lengths of the waveguide L_1 and L_2 are chosen to provide its resonance for the first order longitudinal vibration mode at the frequency $f = 25\,kHz$. For these purposes it is possible to use resonant curves given in [1, 2]. Lengths L_1 and L_2 are related by one-to-one correspondence and it makes it possible to specify only one of these parameters, e.g. L_1, during subsequent tracing of stability diagram. For each resonant configuration of the waveguide described by the first step length L_1 it is necessary to determine critical amplitudes ξ_{cr} of longitudinal vibrations of the waveguide input cross-section corresponding to the loss of stability for different mode orders of flexural vibrations. In other words it is necessary to trace stability diagram on the plane (L_1, ξ_{cr}) for the plurality of flexural vibration modes.

3. Modeling approach and numerical results

3.1 Reduction of the problem to Mathieu's equation

As it was previously shown [1, 2], the most precise technique of modeling flexural vibrations of flexible waveguides is the one based on Timoshenko theory. According to this theory flexural vibrations of the waveguide with a variable cross-section under the action of axial driving force are described by equation

$$\mathbf{D}\cdot\begin{pmatrix}\eta & \alpha\end{pmatrix}^T = 0\,, \qquad (1)$$

where η is the transverse displacement amplitude,
α is the amplitude of cross-section angular displacements.
Differential operator **D** is of the form

$$\mathbf{D} = \begin{pmatrix} \frac{\partial}{\partial x}\left((K_s GS(x) + P(x,t))\frac{\partial}{\partial x}\right) + \rho S(x)\frac{\partial^2}{\partial t^2} & -K_s G\left(\frac{dS}{dx} + S(x)\frac{\partial}{\partial x}\right) \\ K_s GS(x)\frac{\partial}{\partial x} & \frac{\partial}{\partial x}\left(J(x)\left(E - \frac{P(x,t)}{S(x)}\right)\frac{\partial}{\partial x}\right) - K_s GS(x) + \rho J(x)\frac{\partial^2}{\partial t^2} \end{pmatrix}$$

where K_s is the shape factor of the waveguide cross-section,
G is the shear modulus of the waveguide material,
S is the waveguide cross-sectional area,
$P(x,t) = p(x)\cos\Omega t$ is the axial driving force,
p(x) is the driving force amplitude,
Ω is the circular frequency of the driving force,
ρ is the waveguide material density,
J is the centroidal moment of inertia of the waveguide cross-section,
E is the modulus of elasticity of the waveguide material.
According to Bubnov-Galerkin method we will search approximate solution of the Eq. (1) in the form of expansion in terms of natural modes of the waveguide flexural vibrations:

$$\begin{pmatrix}\hat{\eta}(x,t) & \hat{\alpha}(x,t)\end{pmatrix}^T = \sum_m \theta_m(t)\cdot\begin{pmatrix}\varphi_m^{(1)}(x) & \varphi_m^{(2)}(x)\end{pmatrix}^T\,, \qquad (2)$$

where basis vector-functions $\overline{\varphi}_m(x) = \begin{pmatrix}\varphi_m^{(1)}(x) & \varphi_m^{(2)}(x)\end{pmatrix}^T$ are the eigenvectors of the generalized eigenvalue problem:

$$(\mathbf{D}_0 - \omega^2\mathbf{B}(x))\cdot\overline{\varphi} = 0\,, \qquad (3)$$

and coefficients $\theta_m(t)$ of the expansion are unknown functions of time.

Eigenvalues ω_m of the problem (3) correspond to the natural circular frequencies of the *m*th mode of the waveguide flexural vibrations, functions $\varphi_m^{(1)}(x)$ – to the transverse displacement amplitudes for the *m*th mode of the waveguide flexural vibrations, and functions $\varphi_m^{(2)}(x)$ – to the angular displacement amplitudes.

Differential operator $\mathbf{D}_0$ and matrix $\mathbf{B}(x)$ can be derived from the operator **D** under condition $P = 0$ and harmonic law of solutions variation in time ($\partial^2/\partial t^2 \to -\omega^2$) by separating terms comprising natural frequencies ω and can be determined by the following equations:

$$\mathbf{D}_0 = \frac{1}{\rho}\begin{pmatrix} \frac{d}{dx}\left(K_s GS(x)\frac{d}{dx}\right) & -K_s G\left(\frac{dS}{dx} + S(x)\frac{d}{dx}\right) \\ K_s GS(x)\frac{d}{dx} & \frac{d}{dx}\left(EJ(x)\frac{d}{dx}\right) - K_s GS(x) \end{pmatrix},$$

$$\mathbf{B}(x) = \begin{pmatrix} S(x) & 0 \\ 0 & J(x) \end{pmatrix}. \quad (4)$$

It can be shown that operator $\mathbf{D}_0$ is Hermitian, i.e.

$$\int_0^L \bar{u}^T \mathbf{D}_0 \bar{v}\, dx = \int_0^L \bar{v}^T \mathbf{D}_0 \bar{u}\, dx,$$

where vector-functions $\bar{u}$ and $\bar{v}$ satisfy boundary conditions describing clamping of the waveguide ends.

Since operator $\mathbf{D}_0$ is Hermitian and elements of the matrix **B**(x) are non-negative, then eigenvectors of the problem (3) satisfy generalized orthogonality condition [5, part 15.4.6]

$$\int_0^L \bar{\varphi}_m^T \mathbf{B}(x) \bar{\varphi}_n\, dx = 0 \text{ for } m \neq n.$$

With account for the Eqs. (4) this condition can be written in the form:

$$\int_0^L (S(x)\varphi_m^{(1)}\varphi_n^{(1)} + J(x)\varphi_m^{(2)}\varphi_n^{(2)})\, dx = 0.$$

According to the Bubnov-Galerkin method coefficients $\theta_m(t)$ should satisfy the following condition:

$$\int_0^L \bar{\varphi}_n^T \mathbf{D} \cdot \begin{pmatrix} \hat{\eta} & \hat{\alpha} \end{pmatrix}^T dx = 0. \quad (5)$$

Approximation error originating from substitution of exact solution of the Eq. 1 with approximate solution (2) and appearing in the condition (5) is determined by the vector:

$$\mathbf{D}\cdot\begin{pmatrix}\hat{\eta} & \hat{\alpha}\end{pmatrix}^T=\sum_m\begin{pmatrix}(\ddot{\theta}_m+\omega_m^2\theta_m)S(x)\varphi_m^{(1)}+\dfrac{\theta_m\cos\Omega t}{\rho}(p(x)(\varphi_m^{(1)})')' \\ (\ddot{\theta}_m+\omega_m^2\theta_m)J(x)\varphi_m^{(2)}-\dfrac{\theta_m\cos\Omega t}{\rho}\dfrac{d}{dx}\left(\dfrac{p(x)J(x)(\varphi_m^{(2)})'}{S(x)}\right)\end{pmatrix}.$$

Inserting approximation error into the condition (5) we obtain with account for the orthogonality condition equation of the form:

$$\ddot{\theta}_n+\omega_n^2\theta_n+\cos\Omega t\cdot\sum_m c_{nm}\theta_m=0 \qquad (6)$$

with coefficients defined by expression:

$$c_{nm}=\frac{1}{\rho}\left(\int_0^L(S(x)(\varphi_n^{(1)})^2+J(x)(\varphi_n^{(2)})^2)dx\right)^{-1}\left(\int_0^L\sigma(x)J(x)(\varphi_m^{(2)})'(\varphi_n^{(2)})'dx-\right.$$
$$\left.-\int_0^L\sigma(x)S(x)(\varphi_m^{(1)})'(\varphi_n^{(1)})'dx\right),$$

where $\sigma(x)=p(x)/S(x)$ is the amplitude of the axial stresses induced by the action of driving force P(x).

Eq. (6) in the case of single-mode vibrations corresponding to the one term in the expansion (2) takes the form:

$$\ddot{y}+(a-2q\cos 2z)y=0, \qquad (7)$$

where new variables $z=\Omega t/2$, $a=4\omega_n^2/\Omega^2$, $q=-2c_{nn}/\Omega^2$ and $y(z)=\theta_n(2z/\Omega)$ are introduced.

Eq. (7) is known as Mathieu equation. Stability of its solutions depends on the parameters a and q and can be represented in the form of stability regions on the plane (a,q). Graphic representation of the stability regions is known as Ince-Strutt diagram. Boundaries of the stability regions correspond to 2π-periodic solutions of the Eq. (7) and they are determined by characteristic equation which is usually written in the form of continued fraction. In the case $q\to 0$ Eq. (7) takes the form of harmonic oscillations equation which has 2π-periodic solution under condition $a=n^2$, where n is natural number. This implies that for $q\to 0$ boundaries of the stability regions are defined by equation $\Omega\approx 2\omega_n/n$. For $n=1$ (principal instability region) this equation takes the form $\Omega\approx 2\omega_n$, i.e. circular frequency of the driving force should be equal to the doubled natural circular frequency of the waveguide flexural vibrations. More precise equation of boundaries of the principal instability region can be obtained using characteristic equation:

$$\Omega\approx 2\omega_n\sqrt{1\pm q}.$$

3.2 Calculation of parameters of the Mathieu equation

For determination of the parameters *a* and *q* in the Eq. (7) for each resonant configuration of the waveguide it is necessary to calculate natural frequencies and modes of flexural vibrations as well as distribution of the amplitude σ(x) of the axial stresses along the waveguide length. As it was previously shown [1, 2], natural frequencies of vibrations calculated on the basis of Timoshenko theory correspond with high precision (error no more than 0.2 %) to the natural frequencies calculated by means of finite elements method (FEM). In this relation we used ANSYS® software for determination of natural frequencies and modes of vibrations. Accounting for the need of solving sequence of the problems corresponding to the plurality of the waveguide resonant configurations and differing only with geometric parameters process of modeling was automated by means of creation of input listing in APDL (ANSYS Parametric Design) language. The listing consists of plurality of cycles with each cycle corresponding to modeling of single waveguide configuration and including the following main actions:

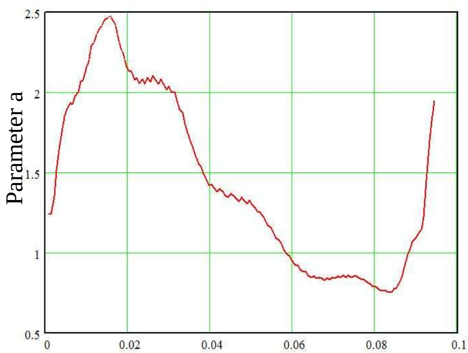
1. Building of the waveguide geometric model and application of boundary conditions.
2. Generation of finite element mesh.
3. Modal analysis of the waveguide.
4. Post-processing of analysis results and calculation of the parameters *a* and *q*.

For each configuration of the waveguide parameters (L_1, L_2) which are necessary for building of its geometric model are read from the file created during tracing of the resonant curve for the longitudinal vibrations using MathCAD® software. Since the problem is symmetric we considered geometric model in the form of two volumes corresponding to the quarters of the waveguide with application of symmetry boundary conditions on the sectional plane. Depending on the type of vibrations under consideration additional boundary conditions were also applied: symmetry boundary conditions for the division plane of the volumes in the case of longitudinal vibrations and constraint on all degrees of freedom for the input cross-section displacements in the case of flexural vibrations. The latter type of boundary conditions led to extraction of both flexural vibrations and quarter-wavelength longitudinal vibrations, which were excluded from consideration during subsequent post-processing. During modal analysis of the longitudinal vibrations first two modes of vibrations corresponding to the waveguide movement as a rigid body and to the first order longitudinal vibration mode were extracted. During modal analysis of the flexural vibrations first ten modes of vibrations were extracted, one of which was identified during post-processing as a quarter-wavelength longitudinal vibration mode of the first order. Natural frequencies f_n of flexural vibrations found as a result of modal analysis were used for calculation of the parameter *a* according to the equation $a = 4 f_n^2 / f^2$. During post-processing we used interpolation of analysis results onto the lines (paths) in the role of which longitudinal axis of the waveguide (path P1) and its generating line (path P2) were chosen. As a result, the so-called path variables on which mathematical operations including differentiation and integration can be performed were formed. During post-processing of results of the longitudinal vibrations modal analysis we used path P1 and axial component u_{x1} of displacement of this path points corresponding to the amplitude ξ of the longitudinal vibrations as the path variable. Numerical differentiation of this path variable with respect to the axial coordinate was used for determination of amplitude of the axial stresses σ according to the equation $\sigma(x) = E\dfrac{d\xi}{dx}$. Unlike the parameter *a* parameter *q* depends both on the length L_1 characterizing waveguide configuration and amplitude ξ(0) of vibrations of the waveguide input cross-section. However, since any change of the amplitude ξ(0) leads to the proportional change of the mechanical stresses amplitude, i.e. $\sigma(x) = \dfrac{E\xi(0)}{u_{x1}(0)}\dfrac{du_{x1}}{dx}$, then parameter *q* will be related to the vibrations amplitude of the input cross-section by linear dependence and it is sufficient to calculate its value $q_0(L_1) = q(L_1, \xi_0)$ for the certain arbitrary value of the amplitude $\xi(0) = \xi_0$, e.g. $\xi_0 = 1\,\mu m$. During post-processing of results of the flexural vibrations modal analysis we used paths P1 and P2. In the role of the first path we used variable component u_{y1} of displacement of the path P1 points orthogonal to the waveguide axis and corresponding to the amplitude η of the flexural vibrations (eigenfunction $\varphi_n^{(1)}$). In the role of the second path variable component u_{x2} of displacement of the path P2 points parallel to the waveguide axis was used. This variable was used for determination of the amplitude α of the waveguide cross-section angular displacements (eigenfunction $\varphi_n^{(2)}$) according to the equation $\alpha(x) \approx 2u_{x2}(x)/d(x)$, where d(x) is the waveguide diameter. Quarter-wavelength longitudinal vibration mode was excluded from consideration by means of checking condition $|u_{x1}(L)/u_{y1}(L)| < 0.1$, where u_{x1} is the component of displacement of the path P1 points parallel to the waveguide axis for the considered vibration mode. This condition is satisfied only for the flexural vibrations. Calculation of the parameter *q* was implemented by means of numerical integration. Values of the parameters *a* and *q* for all nine flexural vibration modes were written for every waveguide configuration into the text file for the subsequent reading by MathCAD® software during tracing of the stability diagram.

3.3 Tracing of the stability diagram

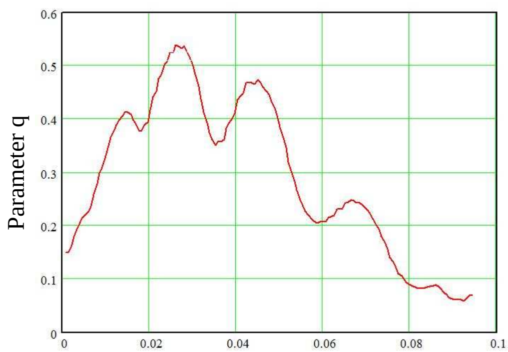
Parameter *a* was determined during tracing of the stability diagram for each value of the length L_1 corresponding to the resonant configuration of the waveguide. Then critical value $q_{cr}(L_1)$ of the parameter *q* corresponding to the parameter *a* was found from the characteristic equation of boundaries of the stability regions. This critical value corresponds to the critical amplitude ξ_{cr} of vibrations of the waveguide input cross-section, i.e. $q_{cr}(L_1) = q(L_1, \xi_{cr})$. Since parameter *q* is related to the vibrations amplitude of the input cross-section by linear dependence then $q(L_1, \xi_{cr})/q(L_1, \xi_0) = \xi_{cr}/\xi_0$, whence it follows $\xi_{cr}(L_1) = \xi_0 q_{cr}(L_1)/q_0(L_1)$.

Fig. 1 provides calculated plots of dependence of the parameters *a* and *q* on the length L_1 for the 8th order flexural vibration mode and Fig. 2 shows plots of the lower and upper boundaries of the instability region corresponding to this mode.



First step length L_1, m

Fig. 1a



First step length L_1, m

Fig. 1b

As it follows from the Fig. 1a, values of the length L_1 equal to 0.0585 and 0.0878 m correspond to the value of the parameter *a* equal to unity. It follows from the Fig. 2a that for these values of the length loss of the waveguide stability occurs for indefinitely small values of the amplitude of the input cross-section vibrations that corresponds to the point (1, 0) on the Ince-Strutt diagram graphed on the plane (a, q). In the real conditions owing to the presence of damping loss of stability will occur only for non-zero values of the amplitude [4].

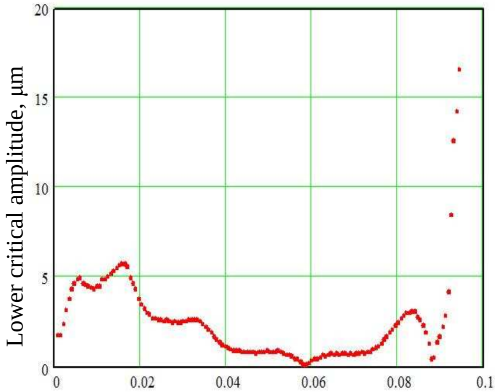


Fig. 2a

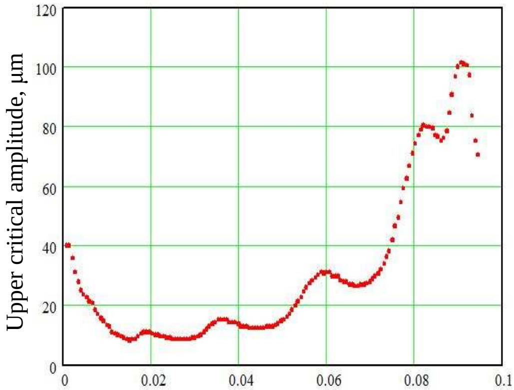


Fig. 2b

4. Conclusion

Developed mathematical model can be used for study of the influence of geometric and structural parameters of the waveguide on its dynamic stability, which is important for development of efficient designs of flexible ultrasonic waveguides for application in technology and medicine.

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