# 585. The research of dynamics of mechanical system with variable parameters

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**Abstract.** Free vibrations of the single-mass system with variable parameters that characterize inertia, friction, rigidity are considered. For an exact analytical solution the method of constant coefficients under derivatives with a new variable is proposed.

**Keywords:** vibrations, variable parameter, differential equation.

## Introduction

As it is known, the main object of dynamics is the definition of the law of the body motion with respect to known forces exerted on it. For the solution of this problem the differential equation should be derived and its integration will lead to the desired law of motion. In the case when the mathematical system has fixed parameters of motion, modern mathematical tool allows convenient fulfillment of the corresponding operations. But, due to onrush of science and engineering, generation of more precise solutions is required. There is a need in consideration of dozens of factors that affect the dynamics of a mechanical system. Therefore it is common that in the considered differential equations the parameters of motion (namely, inertial coefficient, quasi-elastic coefficient, environmental resistance coefficient and so on), are various functions. Thus, there is a necessity to integrate differential equation with variable coefficients.

Currently, there are methods of solution of separate classes of differential equations with variable coefficients, for instance: equation of Euler, Mathieu-Hill equation, equation of Meshcherskoi. Also the formula of Liouville-Ostgorskii, Kulikow method of solution of linear and nonlinear equations with variable coefficients in total derivatives, grid method of solution of linear differential equations with variable coefficients based on Taylor development are known. But these methods are not always acceptable in solution of problems of dynamics as they impose strict limits. And the obtained solution of the Mathieu-Hill equation does not describe the movement characteristics: such as amplitude, oscillation period, speed and acceleration - only stability and instability zones of mechanical system are examined.

**The aim** of this work is to develop the method of analytical solution of a separate class of linear differential equation with variable coefficient.

Let's examine linear homogeneous differential equation of the second kind with variable coefficient and find its terms of integrability

$$a_0(t)\frac{d^2x}{dt^2}+a_1(t)\frac{dx}{dt}+a_2(t)x=0 \qquad (1)$$

Let's choose arbitrary substitution $z = f(t)$ (2)

and name $f(t)$ the defining function. Let's suppose that given function is continuous and defined on the interval $t \in (t_0;t_k)$, twice differentiated and $f'(t) \neq 0$.

Denote $\frac{dz}{dt}=f'(t)=\varphi(t)$. (3)

We need to find the first and the second derivative of the equation (1), we substitute them in the initial equation, and, grouping together the summands with identical derivatives, we present the equation (1) as

$$\left[a_0(t)\varphi^2(t)\right]\frac{d^2x}{dz^2}+\left[a_0(t)\varphi(t)\frac{d\varphi(t)}{dz}+a_1(t)\varphi(t)\right]\frac{dx}{dz}+a_2(t)x=0 \quad (4)$$

The received equation can always be integrated, if coefficients of unknown $x$, $\frac{dx}{dz}$, $\frac{d^2x}{dz^2}$ will be invariable. So let's suppose that

$$a_0(t)\varphi^2(t)=C_0 \quad (5)$$

$$a_0(t)\varphi(t)\frac{d\varphi(t)}{dz}+a_1(t)\varphi(t)=C_1 \quad (6)$$

$$a_2(t)=C_2 \quad (7)$$

where $C_0$, $C_1$, $C_2$ are invariable.

Considering the obtained formula, we can get a new differential equation with constant coefficients, with derivatives with respect to $z$

$$C_0\frac{d^2x}{dz^2}+C_1\frac{dx}{dz}+C_2x=0\,. \quad (8)$$

The solution of the differential equation (8) can be written as

$$x(z)=K_1e^{r_1z}+K_2e^{r_2z}, \text{ or } x(z)=e^{rz}\left(K_1+K_2z\right), \quad (9)$$

in case if the roots of characteristic equation are real numbers $r_1, r_2$ or $r_1=r_2=r$.

If the roots are complex numbers $r_{1,2}=\alpha\pm\beta i$, the solution of equation (8) will be expressed as

$$x(z)=e^{\alpha z}\left(K_1\cos\beta z+K_2\sin\beta z\right) \quad (10)$$

Keeping in mind substitution $z=f(t)$, let's come over to original variable $t$, thus will get the solution of the equation (1)

$$x(t)=K_1e^{r_1f(t)}+K_2e^{r_2f(t)}, \quad x(t)=e^{rf(t)}\left[K_1+K_2f(t)\right]$$

$$\text{or } x(t)=e^{\alpha f(t)}\left\{K_1\cos\left[\beta f(t)\right]+K_2\sin\left[\beta f(t)\right]\right\} \quad (11)$$

where $K_1$ and $K_2$ are constant values, defined from initial data.

Now we need to define the terms of integrability of the equation (1). For this purpose, using the equation (5)-(7) and (3), one can find what coefficient $a_0(t)$, $a_1(t)$, $a_2(t)$ should be

$$a_0(t)=\frac{C_0}{\varphi^2(t)}=\frac{C_0}{\left[f'(t)\right]^2}\,. \quad (12)$$

$$a_1(t)=\frac{C_1}{f'(t)}-\frac{C_0\cdot f''(t)}{\left[f'(t)\right]^3}\,. \quad (13)$$

And, finally, it was previously defined that coefficient $a_2(t)=C_2$. (14)

Substituting obtained equations (12), (13), (14) into initial differential equation (1), one will receive the following

$$\frac{C_0}{\left[f'(t)\right]^2}\frac{d^2x}{dt^2}+\left\{\frac{C_1}{f'(t)}-\frac{C_0\cdot f''(t)}{\left[f'(t)\right]^3}\right\}\frac{dx}{dt}+C_2x=0\,. \tag{15}$$

The above equation represents linear homogeneous differential equation of the second kind with constant coefficients, which always allows analytical integration. Arbitrary constant $C_0$, $C_1$ and $C_2$ take the form of coefficients of equation with new derivative $z$, and $f(t)$ - defining function.

Let's determine sufficient terms of integrability of the initial differential equation with variable coefficients.

From equation (5)-(7) it follows that

$$\begin{cases}\varphi(t)=\sqrt{\dfrac{C_0}{a_0(t)}}\\ a_1(t)=\dfrac{C_1}{\sqrt{C_0}}\sqrt{a_0(t)}+\dfrac{\dot{a}_0(t)}{2}\\ a_2(t)=C_2\end{cases} \tag{16}$$

It follows herefrom that linear differential equation of the second kind with variable coefficient can be integrated in quadratures only if there was a definite dependence between the coefficients $a_0(t)$ and $a_1(t)$ and coefficient $a_2(t)$ was constant (16).

In this case the defining function is determined as

$$f(t)=\int\varphi(t)\,dt+C=\sqrt{C_0}\int\frac{dt}{\sqrt{a_0(t)}}+C\,. \tag{17}$$

Similarly, one can get the terms of integrability for homogeneous linear differential equation with variable coefficient of the third and the fourth kind [1-2].

In order to solve inhomogeneous linear differential equation with variable coefficient one should additionally make a substitution in the right part of the equation $t=f_1(z)$ and solve the obtained equation with the constant coefficients and derivatives with new variable, with the help of commonly known methods.

Linear equation as

$$(ax+b)^n\,y^{(n)}+A_1(ax+b)^{n-1}\,y^{(n-1)}+...+A_{n-1}(ax+b)\,y'+A_ny=f(x)\,, \tag{18}$$

where $a,b,A_1,...,A_n$ are constants and called the equation of Euler. As it is known, for the solution of the equation of such kind one should introduce new independent variable $t$, supposing $ax+b=t$.

We can show that while solving the equation of Euler with the method we introduced, we will come to the same substitution and will get the same result.

**Example 1.** To solve the equation of Euler

$$t^3\frac{d^3x}{dt^3}-t^2\frac{d^2x}{dt^2}+2t\frac{dx}{dt}-2x=t^3$$

**Solution.** Let's check whether the coefficients of the equation correspond to the received terms of integrability. Assume $C_0=1$; $a_0(t)=t^3$.

Therefore $\dot{a}_0(t)=3t^2$; $\ddot{a}_0(t)=6t$.

1) On sufficient condition $a_1(t)=\frac{C_1}{\sqrt[3]{C_0^2}}\sqrt[3]{a_0^2(t)}+\dot{a}_0(t)$.

In our example $a_1(t)=C_1\sqrt[3]{\left(t^3\right)^2}+3t^2=C_1t^2+3t^2$.

As in initial equation $a_1(t)=-t^2$, so $-t^2=C_1t^2+3t^2$

Therefore, $C_1=-4$.

2) On sufficient condition $a_2(t)=\frac{C_2}{\sqrt[3]{C_0}}\sqrt[3]{a_0(t)}+\frac{C_1}{3\sqrt[3]{C_0^2}}\frac{\dot{a}_0(t)}{\sqrt[3]{a_0(t)}}-\frac{\left[\dot{a}_0(t)\right]^2}{9a_0(t)}+\frac{\ddot{a}_0(t)}{3}$

In our case $a_2(t)=C_2\sqrt[3]{t^3}-\frac{4}{3}\frac{3t^2}{\sqrt[3]{t^3}}-\frac{\left[3t^2\right]^2}{9t^3}+\frac{6t}{3}=C_2t-4t-t+2t$.

In initial equation $a_2(t)=2t$, that is why $2t=C_2t-4t-t+2t$

Therefore, $C_2=5$.

3) On sufficient condition $a_3(t)=C_3$.

As by the data $a_3(t)=-2$, that is why $C_3=-2$.

As it is shown, all variable coefficients meet the sufficient terms of integrability (16), where $C_0=1$; $C_1=-4$; $C_2=5$; $C_3=-2$. Let's solve the initial equation by means of the proposed method.

As $a_0(t)=t^3$ and $C_0=1$, then we arrive at

$\varphi(t)=\sqrt[3]{\frac{C_0}{a_0(t)}}$, that $\varphi(t)=\frac{1}{t}$.

Therefore, $f(t)=\int\varphi(t)dt+C=\int\frac{dt}{t}+C=\ln|x|+C$.

Assume $z=f(t)=\ln t$, hence $t=f_1(z)=e^z$.

Now one should come to a new differential equation, but already with constant coefficients and derivatives with respect to $z$.

$$\frac{d^3x}{dz^3}-4\frac{d^2x}{dz^2}+5\frac{dx}{dz}-2x=e^{3z}.$$

So the roots of the characteristic equation of the corresponding homogeneous equation are equal to $r_1=r_2=1$ and $r_3=2$, then the general solution of the homogeneous equation can be written as

$x_1(z)=e^z\left(K_1+K_2z\right)+K_3e^{2z}$.

Complementary function will be searched in the form of

$x_2(z)=Ae^{3z}$.

As its derivatives are equal to

$\dot{x}_2(z)=3Ae^{3z}$; $\ddot{x}_2(z)=9Ae^{3z}$; $\dddot{x}_2(z)=27Ae^{3z}$, then we obtain

$27Ae^{3z}-36Ae^{3z}+15Ae^{3z}-2Ae^{3z}=e^{3z}$

therefore $A=1/4$.

Consequently, the general solution of the differential equation with constant coefficients

$x(z) = e^z (K_1 + K_2 z) + K_3 e^{2z} + 1/4 e^{3z}$.

Considering the fact, that $z = \ln t$, complete integral with variable coefficients $x(t) = t(K_1 + K_2 \ln t) + K_3 t^2 + 1/4 t^3$.

Solving the given equation by Euler method with the help of substitution $t = e^z$, the solution will be identical.

Therefore one can make a conclusion, that the equation of Euler is a subclass of equations that can be solved with the help of proposed method of constant coefficients under derivatives with a new variable.

As it was stated above, the necessity of integration of differential equation with variable coefficients appeared in the context of solution of the main problem of dynamics.

**Example 2.** To study the motion of the mechanical system**,** described by the following differential equation, if it is known, that in the start time the body was shifted out of equilibrium on 0,1 m and left without initial speed.

$$\frac{1}{(3-\sin t)^2}\frac{d^2x}{dt^2} + \left\{\frac{2}{3-\sin t} + \frac{\cos t}{(3-\sin t)^3}\right\}\frac{dx}{dt} + 5x = 0$$

**Solution.** Assume a $a_0(t) = \frac{1}{(3-\sin t)^2}$; $C_0 = 1$.

Then $\dot{a}_0(t) = \frac{2\cos t}{(3-\sin t)^3}$.

1) On sufficient condition

$$a_1(t) = \frac{C_1}{\sqrt{C_0}}\sqrt{a_0(t)} + \frac{\dot{a}_0(t)}{2}$$

In the given problem $a_1(t) = \frac{C_1}{3-\sin t} + \frac{\cos t}{(3-\sin t)^3}$

Under the statement of problem $a_1(t) = \frac{2}{3-\sin t} + \frac{\cos t}{(3-\sin t)^3}$.

Therefore, $C_1 = 2$.

2) On sufficient condition $a_2(t) = C_2$

As by data $a_2(t) = 5$, then $C_2 = 5$.

It is obvious, that all the variable coefficients correspond to the terms of integrability (16), where $C_0 = 1$; $C_1 = 2$; $C_2 = 5$. Let's find the analytical solution of the initial equation according to the proposed method.

As $a_0(t) = \frac{1}{(3-\sin t)^2}$ and $C_0 = 1$, so

$\varphi(t) = \sqrt{\frac{C_0}{a_0(t)}}$, that $\varphi(t) = 3 - \sin t$.

Hence, defining function

$z = f(t) = \int (3 - \sin t)\,dt = 3t + \cos t$.

Let's come to the differential equation with constant coefficients and derivatives with respect to $z$

$$\frac{d^2x}{dz^2}+2\frac{dx}{dz}+5x=0.$$

The roots of the characteristic equation are equal to $r_{1,2}=-1\pm 2i$. That is why the general solution can be written as

$$x(z)=e^{-z}\left(K_1\cos 2z+K_2\sin 2z\right).$$

or $x(z)=e^{-z}a\sin(2z+\gamma)$,

where $a$, $\gamma$ are constant values, detected out of initial data

$$x\big|_{t=0}=0{,}1,\ \left.\frac{dx}{dt}\right|_{t=0}=0.$$

Considering the fact that $z=3t+\cos t$, we will receive that the law of motion of required mechanical system can be written as

$$x(t)=0{,}11e^{-(3t+\cos t)}\sin\left[2\left(3t+\cos t\right)+1{,}1\right].$$

Given equation describes free attenuation oscillation of the system. Graphic of such oscillation is illustrated in Fig. 1.

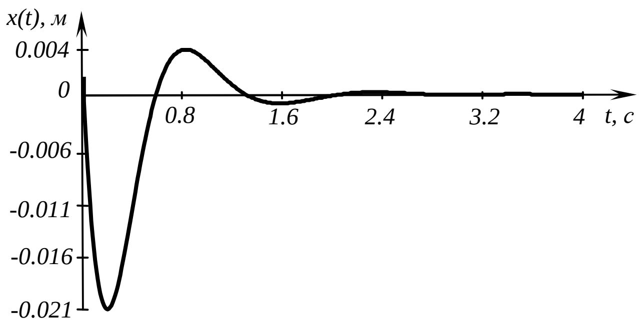


**Fig. 1.** Graphics of free oscillation of the system with variable parameters

## Conclusions

1. In this work the method of constant coefficients under derivatives with a new variable is proposed, which allows integration in quadratures of separate class of the linear differential equations with variable coefficients.
2. Sufficient terms of integrability (16) are established by means of the developed method of differential equations of the second, the third and the fourth kind.
3. Practical application of the method is presented in terms of research of dynamics of the mechanical system.

## References

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