618. Analytical solutions to nonlinear mechanical oscillation problems

M. Ghadimi [1], H. D. Kaliji [1], A. Barari[2]
[1]*Department of Mechanical Engineering, Islamic Azad University, Semnan Branch, Semnan, Iran*
[2]*Department of Civil Engineering, Aalborg University, Sohngårdsholmsvej 57, 9000 Aalborg, Aalborg, Denmark*
E-mail: *ab@civil.aau.dk, amin78404@yahoo.com*

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Abstract. In this paper the Max-Min Method is utilized for solving the nonlinear oscillation problems. The proposed approach is applied to three systems with complex nonlinear terms in their motion equations. By means of this method the dynamic behavior of oscillation systems can be easily approximated using He Chengtian's interpolation. The comparison of the obtained results from Max-Min method with time marching solution and the results achieved from literature verifies its convenience and effectiveness. It is predictable that He's Max-Min Method will find wide application in various engineering problems as indicated in the following cases.

Keywords: Nonlinear vibration, Max-Min method, periodic solution, approximate frequency, time marching solution.

1. Introduction

Oscillation systems have been widely used in many areas of physics and engineering. These systems have significant importance in engineering particularly in mechanical and structural dynamics because there are many practical engineering components consisting of vibrational elements that can be modeled using oscillatory systems such as elastic beams supported by two springs or mass-on-moving belt or nonlinear pendulum and vibration of a milling machine [1-3]. However, most of the oscillation systems are nonlinear, hence analyzing these equations, due to limitation of existing exact solutions, has been one of the most time-consuming and difficult affairs among researchers dealing with vibrations. There is a large variety of approximate methods for solving of nonlinear problems in dynamical systems including Variational Iteration Method [4-6], Homotopy Perturbation Method (HPM) [7-10], Energy Balance Method (EBM) [11-13], Max-Min Method [14,15], etc.

The semi-analytical approach as Max-Min has been exploited to obtain periodic solution of some types of oscillation systems. The method has been primarily proposed based on ancient Chinese mathematics [16,17]. In most engineering problems, it is easy to obtain maximum and minimum thresholds for solution of a nonlinear equation. The comprehensive explanation of the solution procedure of the method and its implication are presented through some examples [14,15]. Also, the accuracy of the method and its proofs are described for different problems in literature [14-18].

In this paper Max-Min Method is applied to three cases addressing the nonlinear motion equation of oscillation system, Duffing equation as well as transverse vibration of the Euler-Bernoulli beam subjected to an axial load, respectively. The results are verified against time marching solution and the ones available within the literature. It will be demonstrated that approximate solutions obtained by means of this method are in excellent agreement with solution from the former method.

2. Application of Max-Min Method

In order to assess the advantages and the accuracy of the Max-Min Method, we will consider the following three cases:

2.1. Case 1

Consider the motion of mass m attached to the centre of a stretched elastic wire with coefficient of stiffness equal to k, see Fig. 1. The length of the elastic wire when no force is applied to it is $2a$. We assume that the movement of the particle is one-dimensional and this is constrained to move only in the horizontal x direction. The equation of motion is given by the following nonlinear differential equation [2, 19]:

$$m\frac{d^2x}{dT^2}+2kx-\frac{2kax}{\sqrt{d^2+x^2}}=0, \tag{1}$$

with initial conditions:

$$x(0)=x_0 \quad and \quad \frac{dx}{dT}(0)=0. \tag{2}$$

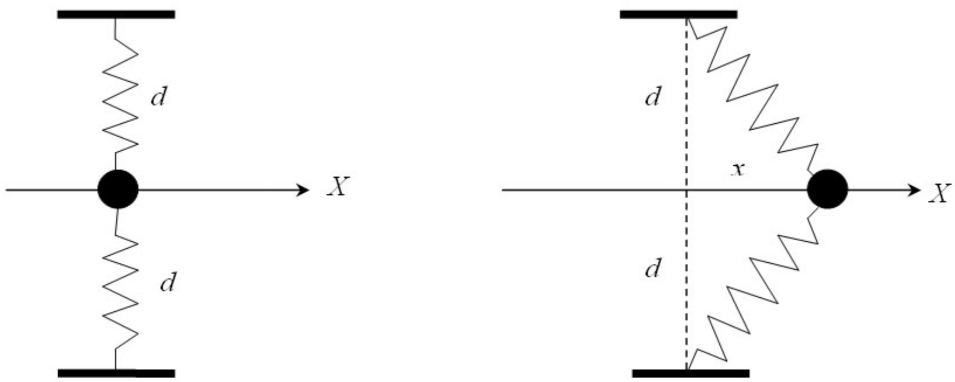

Fig. 1. Mass attached to a stretched wire

Two dimensionless variables u and t can be constructed as follows:

$$u=x/d \quad and \quad t=\sqrt{\frac{2k}{m}}T. \tag{3}$$

Substituting these dimensionless variables into Eq. (1) gives:

$$\frac{d^2u}{dt^2}+u-\frac{\lambda u}{\sqrt{1+u^2}}=0, \tag{4}$$

and initial conditions are:

$$u(0)=A \quad and \quad \frac{du}{dt}(0)=0. \tag{5}$$

In Eqs. (4) and (5) we have defined the following parameters:

$$A=\frac{x_0}{d} \quad and \quad \lambda=\frac{a}{d}. \tag{6}$$

As $0\le a\le d$, it follows that $0<\lambda\le 1$.

According to the initial conditions, we choose a trial function in the form of the following equation which can satisfy the initial conditions:

$$u = A\cos\omega t, \tag{7}$$

where ω is the frequency. For small amounts of A, we have:

$$\frac{d^2u}{dt^2} + u - \frac{\lambda u}{\sqrt{1+u^2}} = 0, \quad \cong \frac{d^2u}{dt^2} + u - \lambda u\left(1 - \frac{u^2}{2}\right) = 0. \tag{8}$$

Then

$$-\omega^2 u + u - \lambda u\left(1 - \frac{u^2}{2}\right) = 0, \quad \Rightarrow \quad \omega^2 = 1 - \lambda\left(1 - \frac{u^2}{2}\right). \tag{9}$$

So, we have:

$$0 < \omega^2 < 1 - \lambda\left(1 - \frac{A^2}{2}\right). \tag{10}$$

According to He Chengtian's interpolation [14], we have:

$$\omega^2 = \frac{m(0) + n(1 - \lambda(1 - A^2/2))}{m+n} = k\left[1 - \lambda\left(1 - \frac{A^2}{2}\right)\right], \tag{11}$$

where m and n are weighting factors, $k=n/(m+n)$; so the frequency and approximate solution can be written as:

$$\omega = \sqrt{k(1 - \lambda(1 - (A^2/2))}, \qquad u(T) = A\cos\left(\left[k(1 - \lambda(1 - A^2/2))\right]^{1/2} .T\right). \tag{12}$$

We rewrite the Eq. (8) by using the term:

$$k\left[1 - \lambda\left(1 - \frac{A^2}{2}\right)\right]u. \tag{13}$$

Hamiltonian is constructed as below:

$$H = \int f(u, \frac{d^2u}{dt^2})du = \int f(u, \frac{d^2u}{dt^2})\cos\omega t\, dt. \tag{14}$$

Now, we can write:

$$B = \int_0^{\tau/4}\left(k\left[1 - \lambda\left(1 - \frac{A^2}{2}\right)\right]u - u + \lambda u\left(1 - \frac{u^2}{2}\right)\right)\cos\omega t dt = 0. \tag{15}$$

By substituting Eq. (12) into Eq. (15), where $\tau = 2\pi/\omega$, we obtain:

$$\begin{cases} B_1 = \int_0^{\tau/4}\left(k\left[1 - \lambda\left(1 - \frac{A^2}{2}\right)\right] - 1 + \lambda\right)A\cos^2\omega t dt, \\ B_2 = \int_0^{\tau/4} -0.5\lambda A^3\cos^4\omega t dt. \end{cases} \tag{16}$$

Because of the two terms in the above integral equation, we solve them one by one. So assume $B = B_1 + B_2$:

$$\begin{cases} B_1 = \left(k\left[1-\lambda\left(1-\dfrac{A^2}{2}\right)\right]-1+\lambda\right)\left(\dfrac{\pi A}{4}\right), \\ B_2 = \left(-\dfrac{\lambda A^3}{2}\right)\left(\dfrac{3\pi}{16}\right), \end{cases} \qquad B = B_1 + B_2 = 0. \tag{17}$$

So, we have:

$$k = \frac{1-\lambda+\frac{3}{8}\lambda A^2}{1-\lambda\left(1-\frac{A^2}{2}\right)}. \tag{18}$$

The approximate Frequency and the solution are evaluated from Eq. (12):

$$\omega_{\max-\min} = \sqrt{1-\lambda+\frac{3}{8}\lambda A^2}, \tag{19}$$

$$u_{\max-\min}(t) = A\cos\sqrt{1-\lambda+\frac{3}{8}\lambda A^2}\; t. \tag{20}$$

To show the remarkable accuracy of the obtained result, we write harmonic balance frequency [20] and compare the approximate periodic solutions with Runge-Kutta 4th order in Table 1 and Fig. 2. In Fig. 3 frequencies obtained by Max-Min Method are compared with those of Homotopy Perturbation Method:

$$\omega_{hbm} = \sqrt{1-\lambda(1+0.75A^2)^{-0.5}}. \tag{21}$$

Table 1. Comparison between Max-Min Method with HBM and time marching solutions for the motion equation (8), when $\lambda = 0.5$ and t = 1(s).

A	Runge-Kutta	Max-Min	Harmonic Balance
0.1	0.075919071	0.075938343	0.075938824
0.2	0.151224256	0.151360321	0.151375427
0.3	0.225387308	0.225751238	0.225862397
0.4	0.298024455	0.298599735	0.299048702
0.5	0.368915818	0.369399444	0.370700183
0.6	0.437990273	0.437650634	0.440698758
0.7	0.505290595	0.502861833	0.509025932
0.8	0.570933911	0.564551436	0.575738449
0.9	0.635077221	0.622249279	0.640942900
1	0.697891859	0.675498188	0.704773650

2.2. Case 2

In this case we have Duffing equation with constant coefficient as follows (see Fig. 4):

$$\ddot{u} + \frac{k_1}{m}u + \frac{k_2}{2mh^2}u^3 = \frac{F_0}{m}\sin\omega_0 t, \qquad u(0) = A, \quad \dot{u}(0) = 0. \tag{22}$$

The following trial function could be assumed, which satisfies the above mentioned initial conditions:

$$u(t) = A\cos\omega t, \qquad \ddot{u}(t) = -\omega^2 u(t). \tag{23}$$

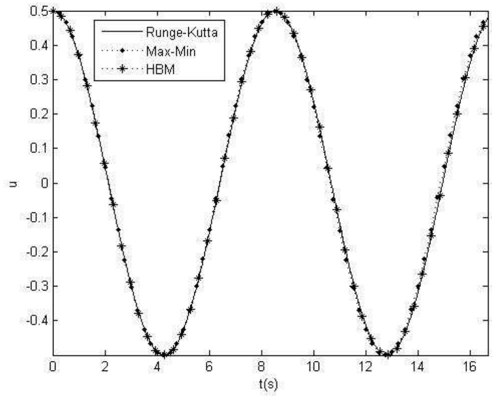

Fig. 2. The comparison of Max-Min solution with Runge-Kutta 4th order and Harmonic Balance solutions for $\lambda = 0.5$ and $A = 0.5$

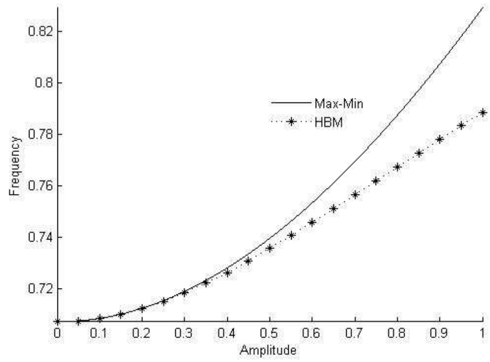

Fig. 3. The comparison between Max-Min and Harmonic Balance Methods, when $\lambda = 0.5$

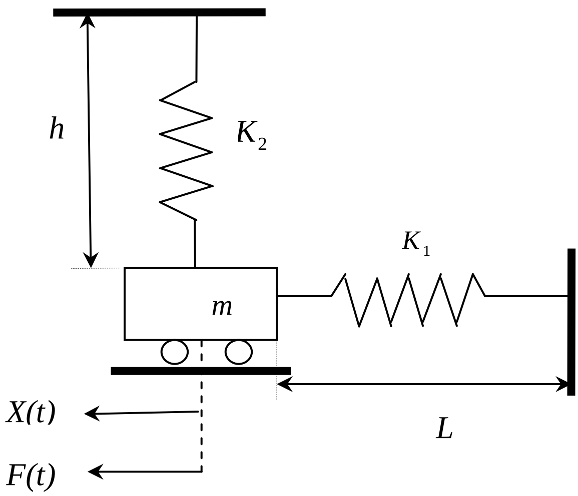

Fig. 4. The physical model of Duffing equation with constant coefficient [21]

Substituting Eq. (23) into Eq. (22) we have:

$$-\omega^2 u + \frac{k_1}{m} u + \frac{k_2}{2mh^2} u^3 = \frac{F_0}{m} \sin \omega_0 t, \tag{24}$$

$$-\omega^2+\frac{k_1}{m}+\frac{k_2}{2mh^2}u^2=\frac{F_0}{m}\sin\omega_0 t(2-u). \tag{25}$$

We rewrite Eq. (25) in the form of:

$$\omega^2=\frac{k_2}{2mh^2}u^2+\frac{F_0\sin\omega_0 t}{m}u+\frac{k_1}{m}-\frac{2F_0\sin\omega_0 t}{m}. \tag{26}$$

According to He Chengtian's interpolation [14], we have:

$$\frac{0}{1}<\omega^2<\frac{\dfrac{k_2A^2}{2mh^2}+\dfrac{F_0A\sin\omega_0 t+k_1-2F_0\sin\omega_0 t}{m}}{1}. \tag{27}$$

So, following the solution routine and using the max-min of frequency, we have:

$$\omega^2=k\left(\frac{k_2A^2}{2mh^2}+\frac{F_0A\sin\omega_0 t+k_1-2F_0\sin\omega_0 t}{m}\right), \tag{28}$$

where m and n are weighting factors, $k=n/(m+n)$.

So the frequency can be approximated as:

$$\omega=\sqrt{k\left(\frac{k_2A^2}{2mh^2}+\frac{F_0A\sin\omega_0 t+k_1-2F_0\sin\omega_0 t}{m}\right)}. \tag{29}$$

And the approximate solution would be:

$$u(t)=A\cos\left(\sqrt{\frac{k_2A^2}{2mh^2}+\frac{F_0A\sin\omega_0 t+k_1-2F_0\sin\omega_0 t}{m}}\;t\right). \tag{30}$$

We rewrite Eq. (24) as we did in the previous case:

$$\begin{aligned}\ddot{u}+k\left(\frac{k_2A^2}{2mh^2}+\frac{F_0A\sin\omega_0 t+k_1-2F_0\sin\omega_0 t}{m}\right)u&=-\frac{k_1}{m}u-\frac{k_2}{2mh^2}u^3+\frac{F_0\sin\omega_0 t}{m}\\&+k\left(\frac{k_2A^2}{2mh^2}+\frac{F_0A\sin\omega_0 t+k_1-2F_0\sin\omega_0 t}{m}\right)u.\end{aligned} \tag{31}$$

Now, we can write:

$$B=\int_0^{T/4}\left(\begin{array}{l}\dfrac{F_0\sin\omega_0 t-k_1u}{m}-\dfrac{k_2u^3}{2mh^2}\\+k[\dfrac{k_2A^2}{2mh^2}+\dfrac{F_0A\sin\omega_0 t+k_1-2F_0\sin\omega_0 t}{m}]\end{array}\right)\cos\omega t dt=0, \tag{32}$$

where $T=2\pi/\omega$ and $u(t)=A\cos(\omega t)$. According to the previous case we have:

$$k=\frac{\dfrac{\pi Ak_1}{4m}+0.1875\dfrac{\pi k_2A^3}{2mh^2}-\dfrac{F_0\sin\omega_0 t}{m}}{0.25\pi A\left(\dfrac{k_1}{m}+\dfrac{k_2A^2}{2mh^2}+\dfrac{F_0(A-2)\sin\omega_0 t}{m}\right)}. \tag{33}$$

So the frequency is obtained as follows:

$$\omega = \sqrt{\frac{\dfrac{\pi A k_1}{4m} + 0.1875\dfrac{\pi k_2 A^3}{2mh^2} - \dfrac{F_0 \sin \omega_0 t}{m}}{0.25\pi A\left(\dfrac{k_1}{m} + \dfrac{k_2 A^2}{2mh^2} + \dfrac{F_0 (A-2)\sin \omega_0 t}{m}\right) \cdot \left(\dfrac{k_2 A^2}{2mh^2} + \dfrac{F_0 A \sin \omega_0 t + k_1 - 2F_0 \sin \omega_0 t}{m}\right)}}. \tag{34}$$

The energy balance frequency for the periodic solution of Eq. (22) is [22]:

$$\omega_{ebm} = \frac{2}{A}\sqrt{\frac{k_1 A^2}{4m} + \frac{3k_2 A^4}{32mh^2} + \frac{F_0 A \sin \omega_0 t}{m}\left(\frac{\sqrt{2}}{2} - 1\right)}. \tag{35}$$

In Table 2 and Fig. 5 for the following parameters we compare the approximate periodic solutions with EBM and Runge-Kutta 4th order, which obviously indicate the accuracy of the results. Also, the obtained frequencies by Max-Min Method are compared with Energy Balance Method (Fig. 6):

$L = 1m,\ h = 0.9m,\ m = 10kg,\ k_1 = 1000N/m,\ k_2 = 1100N/m,\ F_0 = 1N,$

$\omega_0 = 1rad/s.$

Table 2. Comparison between Max-Min Method with EBM and time marching solutions for the motion equation (22), when *t = 1(s)*.

A	Runge-Kutta	Max-Min	Energy Balance
0.1	-0.081575101	-0.085404807	-0.085181061
0.2	-0.154875342	-0.159214473	-0.158958344
0.3	-0.207083274	-0.212350772	-0.212059076
0.4	-0.222283390	-0.228816907	-0.228491006
0.5	-0.182174962	-0.189217954	-0.188870436
0.6	-0.071101744	-0.075158658	-0.074814682
0.7	0.117410419	0.122530263	0.122836268
0.8	0.375215155	0.393070003	0.393303053
0.9	0.673046721	0.696204030	0.696341539
1	0.947453243	0.959562796	0.959607576

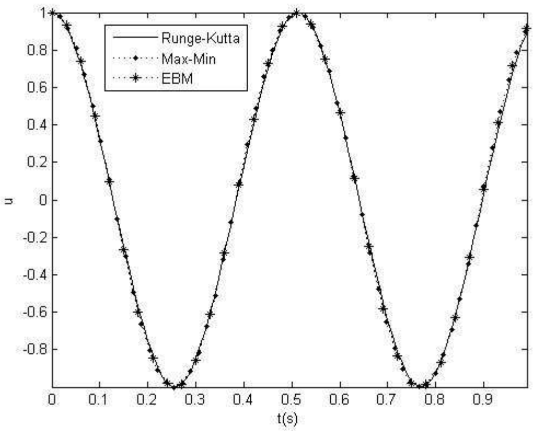

Fig. 5. The comparison of Max-Min solution with Runge-Kutta 4th order and Energy Balance solutions for *A = 1*

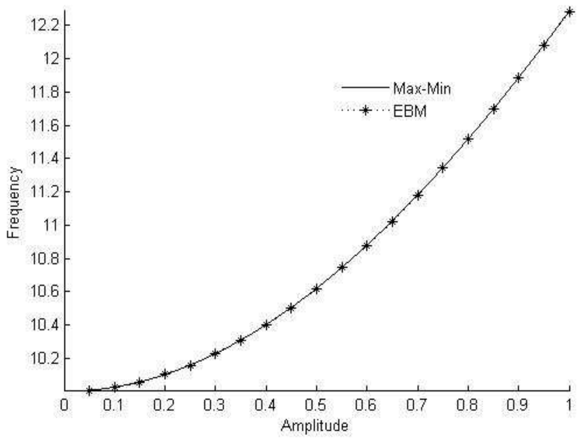

Fig. 6. The frequency comparison between Max-Min Method with Energy Balance Method, when t = 1(s)

2.3. Case 3

Consider a straight Euler-Bernoulli beam subjected to an axial load as illustrated in Fig. 7. The motion equation of nonlinear vibration of the beam is obtained as follows [23]:

$$\ddot{w}(t)+(\alpha_1+p\alpha_2)w(t)+\alpha_3 w^3(t)=0, \tag{36}$$

where α_1, α_2 and α_3 are constant coefficients [23] and p is a non-dimensional axial force.

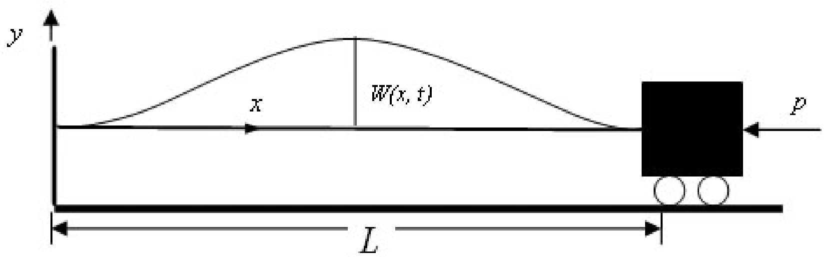

Fig. 7. A schematic of an Euler-Bernoulli beam subjected to an axial load

The center of the beam is subjected to the fallowing initial conditions:

$$w(0)=A, \qquad \dot{w}(0)=0, \tag{37}$$

where A denotes the non-dimensional maximum amplitude of oscillation. We consider the trial function, which satisfies the initial conditions (37):

$$w(t)=A\cos(\omega t) \quad \Rightarrow \quad \ddot{w}(t)=-\omega^2 A\cos(\omega t). \tag{38}$$

According to the trial function, we have:

$$-\omega^2 w+(\alpha_1+p\alpha_2)w+\alpha_3 w^3=0, \quad \Rightarrow \quad \omega^2=\alpha_1+p\alpha_2+\alpha_3 w^2. \tag{39}$$

So, we have:

$$0 < \omega^2 < \alpha_1 + p\alpha_2 + \alpha_3 A^2. \tag{40}$$

According to He Chengtian's interpolation [14], we obtain:

$$\omega^2 = \frac{m(0) + n(\alpha_1 + p\alpha_2 + \alpha_3 A^2)}{m+n} = k(\alpha_1 + p\alpha_2 + \alpha_3 A^2). \tag{41}$$

So, the frequency and approximate solution can be written as:

$$\begin{aligned} &\omega = \sqrt{k(\alpha_1 + p\alpha_2 + \alpha_3 A^2)}, \\ &w(t) = A\cos\left(\left[k(\alpha_1 + p\alpha_2 + \alpha_3 A^2)\right]^{0.5} .t\right). \end{aligned} \tag{42}$$

By rewriting Eq. (36) and subtracting the term $k\,(\alpha_1 + p\alpha_2 + \alpha_3 A^2)w$:

$$\begin{aligned} &f(w,\ddot{w}) = \ddot{w} + k(\alpha_1 + p\alpha_2 + \alpha_3 A^2)\, w \\ &= k(\alpha_1 + p\alpha_2 + \alpha_3 A^2)\, w - (\alpha_1 + p\alpha_2)\, w - \alpha_3 w^3. \end{aligned} \tag{43}$$

Hamiltonian is constructed as below:

$$H = \int f(w,\ddot{w})dw = \int f(w,\ddot{w}) \cos \omega t dt. \tag{44}$$

Now, we can write:

$$\text{B}=\int_0^{\tau/4} \left[k(\alpha_1 + p\alpha_2 + \alpha_3 A^2)\, w - (\alpha_1 + p\alpha_2)\, w - \alpha_3 w^3\right] \cos \omega t\, dt = 0. \tag{45}$$

By substituting Eq. (42) into Eq. (45), where $\tau = 2\pi/\omega$, it is obtained as:

$$\begin{aligned} &B_1 = \int_0^{\tau/4} \left[k(\alpha_1 + p\alpha_2 + \alpha_3 A^2) - (\alpha_1 + p\alpha_2)\right] A\cos^2 \omega t dt, \\ &B_2 = \int_0^{\tau/4} -\alpha_3 A^3 \cos^4 \omega t dt. \end{aligned} \tag{46}$$

Therefore:

$$k = \frac{\alpha_1 + p\alpha_2 + 0.75\alpha_3 A^2}{\alpha_1 + p\alpha_2 + \alpha_3 A^2}. \tag{47}$$

The approximate frequency and deflection are evaluated from Eq. (42):

$$\begin{aligned} &\omega_{\max-\min} = \sqrt{\alpha_1 + p\alpha_2 + 0.75\alpha_3 A^2}, \\ &w(t) = A\cos\left(\left[\alpha_1 + p\alpha_2 + 0.75\alpha_3 A^2\right]^{0.5} \cdot t\right). \end{aligned} \tag{48}$$

The Homotopy Analysis Method (HAM) for the periodic solution of Eq. (36) is [23]:

$$w(t)_{ham} = \frac{A}{32\omega^2}\left(\alpha_3 A^2 + 32\omega^2\right)\cos \omega t - \frac{\alpha_3}{32\omega^2} A^3 \cos 3\omega t. \tag{49}$$

The comparison of the Max-Min Method, Homotopy Analysis Method [23] and Runge-Kutta 4th order solution is given in Table 3 and Fig. 8. The obtained results demonstrate the excellent accuracy of the proposed method.

Table 3. Comparison of results obtained by means of Max-Min Method with the HAM and time marching solutions for the equation (36), when $\alpha_1 = \alpha_2 = \alpha_3 = 1$, *p = 1* and *t = 3* (s)

A	Runge-Kutta	Max-Min	HAM
0.1	-0.044533974	-0.0445560984	-0.0445783342
0.2	-0.084666511	-0.0848343213	-0.0850056779
0.3	-0.115967829	-0.1164794128	-0.1170176727
0.4	-0.134005588	-0.1350238143	-0.1361524778
0.5	-0.134460788	-0.1359318489	-0.1377302081
0.6	-0.113336434	-0.1147486644	-0.1169402170
0.7	-0.067222664	-0.0673645350	-0.0690911941
0.8	0.006445098	0.009612363	0.009922395
0.9	0.109194047	0.118399631	0.122917557
1	0.241402379	0.259542274	0.270544953
1.5	1.252951546	1.299991637	1.324670403

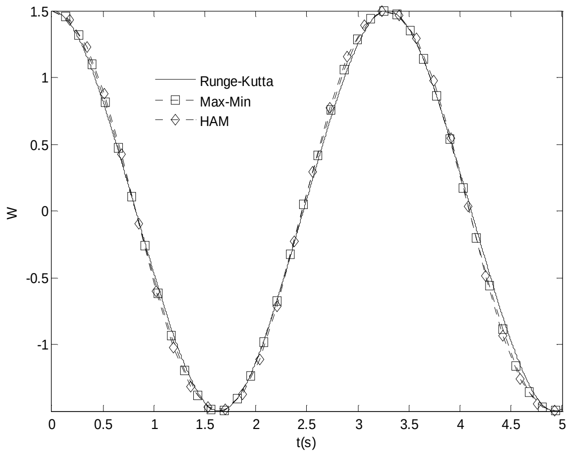

Fig. 8. The comparison of Max-Min solution with Runge-Kutta 4th order and Homotopy Analysis solutions for $\alpha_1 = \alpha_2 = \alpha_3 = 1$, *p = 1* and *A = 1.5*

3. Conclusion

In this study we investigated the Max-Min Method to solve nonlinear governing equations of the oscillators and vibrations. This method was applied to three cases with different type of nonlinearity to demonstrate the effectiveness and efficiency of the technique. Comparison of the obtained results with the time marching solution indicates that Max-Min Method is a reliable technique to approximate the periodic solution of the oscillatory systems for the small amplitude. However, further research is needed in order to develop the Max-Min Method for other applications in engineering.

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