703. A semi-analytical approach for the response of nonlinear conservative systems

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Abstract. This work applies Parameter expanding method (PEM) as a powerful analytical technique in order to obtain the exact solution of nonlinear problems in the classical dynamics. Lagrange method is employed to derive the governing equations. The nonlinear governing equations are solved analytically by means of He's Parameter expanding method. It is demonstrated that one term in series expansion is sufficient to generate a highly accurate solution, which is valid for the whole domain of the solution and system response. Comparison of the obtained solutions with the numerical ones indicates that this method is an effective and convenient tool for solving these types of problems.

Keywords: Lagrange method, nonlinear dynamics, parameter expanding method.

I. Introduction

A series of research works have appeared within the literature in recent years discussing the steady forced oscillations of mechanical systems with different degrees of freedom. The stability of linear and nonlinear nonautonomous single-degree-of-freedom (SDOF) systems loaded with nonconservative forces is analyzed by Glabisz [1]. In his simulations for linear case, analytical expressions were generated for the boundaries of the stability regions. Later, Morison suggested SDOF techniques for dynamic modeling of reinforced concrete flexural elements under blast as well as ground-shock loadings [2]. Morison concluded that the equivalent SDOF approach is an appropriate alternative solution, but the most published parameters for two-way spanning members are inaccurate by up to 50% for several parameters [2]. Yamgoue and Kofane [3] also generalized a developed perturbation method for conservative SDOF systems subjected to the damping forces. They reported explicit solution as a function of amplitude, frequency and phase of oscillation by merging the classical Krylov-Bogoliubov-Mitropolsky method and a modified Lindstedt-Poincare method.

The analysis of nonconservative systems has attracted the attention of many workers recently. The main concern within their research works is the stability of nonconservative autonomous systems under static loads dependent on only the state of displacement. The representative works in the field of analysis of nonconservative systems are those by Ziegler [4], Bolotin [5], Prasad and Herrmann [6], Zyckowski [7] and Kounadis [8].

Also, most common problems in the oscillation systems are inherently nonlinear. Except a limited number of these problems, most of them do not have analytical solutions. Therefore, these nonlinear equations should be analyzed using alternative solutions such as numerical techniques and perturbation methods [9]. In the numerical methods, stability and convergence criteria should be considered to avoid divergence or inappropriate results. In the perturbation methods, a small parameter is inserted in the equation. Therefore, finding the small parameter and inserting it into the equation are deficiencies of these methods.

Recently, considerable attention has been directed towards analytical solutions for nonlinear equations without small parameters. Many new techniques have appeared in the literature, for example, the homotopy perturbation method [10–14], homotopy analysis method [15–18], the variational iteration method [19–21], and the energy balance method [22-24].

He's Parameter expanding method (PEM) is the most effective and convenient method to solve nonlinear differential equations analytically. It is shown that HPEM is another efficient method which is able to solve a large class of linear and nonlinear problems with components that converge rapidly to accurate solutions easily and accurately. HPEM was first proposed by He and then has been successfully applied to various engineering problems [25-28].

There are a few works on using Parameter expanding method in the literature. He in [25] proposed modified Lindstedt–Poincare method for some strongly nonlinear oscillations. Liu [26] studied approximate period of nonlinear oscillators with discontinuities by modified Lindstedt–Poincare method. Xu [27] suggested He's parameter-expanding method for strongly nonlinear oscillators, while Tao [28], proposed frequency-amplitude relationship of nonlinear oscillators using He's Parameter expanding method.

In this study He's Parameter expanding method is used to investigate the behavior of nonlinear problems in dynamics. To show the accuracy and applicability of this method some examples are studied and compared with numerical methods. To obtain the governing equations, Lagrange method is utilized. Some remarkable virtues of the methods are studied, and their applications to obtain the higher-order approximate periodic solutions are illustrated. By using simultaneously Lagrange and PEM method, it seems very easy to study the behavior of dynamical systems and also calculate the natural frequency and limit cycle.

2. Lagrange equation:

A differential equations of motion expressed in terms of generalized coordinates is called Lagrange equation [9]. Lagrange equation including nonconservative forces is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}=Q_i \qquad i=1,2,...N \tag{1}$$

3. He's Parameter expanding method

In case no parameter exists in an equation, HPEM can be used. As a general example, it can be considered the following equation:

$$m\,x''+\omega_0^2 x+\eta\, f\,(x,x',x'')=0, \qquad x(0)=A, \qquad x'(0)=0 \tag{2}$$

According to the bookkeeping parameter method, the solution is expanded into a series of p in the form:

$$x(t)=u_0(t)+p\,x_1(t)+p^2 x_2(t)+\ldots \tag{3}$$

Hereby the parameter p does not require being small $0\le p\le\infty$.

The coefficients m and ω_0^2 are expanded in a similar way:

$$m=1+p\,m_1+p^2 m_2+\ldots \tag{4}$$

$$\omega_0^2=\omega^2+p\,\omega_1+p^2\omega_2+\ldots \tag{5}$$

$$\eta=p\,c_1+p^2 c_2+\ldots \tag{6}$$

ω is assumed to be the frequency of the studied nonlinear oscillator, the values for m and ω_0^2 can be any positive, zero or negative real value.

Here, we are going to solve some problems by using He's parameter expanding method.

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4. Case 1:

4. 1. Motion of a particle on a rotating parabola

An example of a SDOF conservative system has been considered that is described by an equation as follows. The motion of a ring of mass *m* sliding freely on the wire described by the parabola $z = rx^2$, which rotates with a constant angular velocity Ω about the *z*-axis as shown in Fig. 1. It is convenient to write the equation of motion of the ring by using a Lagrange formulation. For a conservative, Holonomic system, the kinetic and potential energies *T* and *V* can be expressed in terms of so-called generalized coordinate *q*, where *q* is a vector whose elements are the independent coordinate needed to describe the system under consideration. For the present problem the kinetic and potential energies are:

$$T = \frac{1}{2} m(x'^2(t) + \Omega^2 x^2(t) + z'(t)^2) \tag{7}$$

$$V = mgz(t) \tag{8}$$

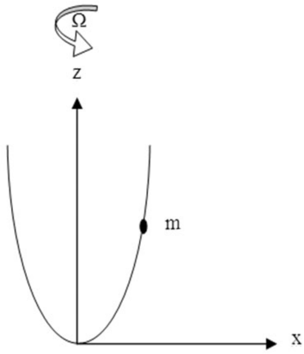

Fig. 1. The geometry of example (1)

Using the concentrate $z = rx^2$, the above equations can be rewritten as:

$$T = \frac{1}{2} m((1 + 4r^2 x^2(t))x'^2(t) + \Omega^2 x^2(t)) \tag{9}$$

$$V = mgrx^2(t) \tag{10}$$

Substituting *T* and *V* from Eqs. (9) and (10) into Lagrange equation yields:

$$L = \frac{1}{2} m[(1 + 4r^2 x^2(t))x'^2(t) + \Omega^2 x^2(t)] - mgrx^2(t) \tag{11}$$

Finally the equation of motion becomes:

$$(1 + 4r^2 x^2(t))x'^2(t) + Ax(t) + 4r^2 rx'^2(t)x(t) = 0, \tag{12}$$

where *A* is:

$$A = 2gr - \Omega^2 \tag{13}$$

4. 2. Application of HPEM

According to the PEM, Eq. (12) can be rewritten as:

$$1\frac{d^2 x(t)}{dt^2} + A\,x(t) - 4r^2[(x^2(t)x''(t) + x'^2(t)x(t))] = 0 \tag{14}$$

with the following initial conditions:

$$x(0) = \lambda, \qquad x'(0) = 0. \tag{15}$$

The form of solution and the constant one in Eq. (14) can be expanded as:

$$x(t) = x_0(t) + p\,x_1(t) + p^2 x_2(t) + \ldots \tag{16}$$

$$1 = 1 + pa_1 + pa_2 + \ldots \tag{17}$$

$$A = \omega^2 + pb_1 + pb_2 + \ldots \tag{18}$$

$$4r^2 = pc_1 + pc_2 + \ldots \tag{19}$$

Substituting Eqs. (16) through (19) into Eq. (14), and applying the standard perturbation method, we have:

$$x_0''(t) + \omega^2 x_0(t) = 0, \qquad x_0(0) = \lambda, \qquad x_0'(0) = 0 \tag{20}$$

$$\frac{d^2 x_1(t)}{dt^2} + c_1 x_0{}^2(t)\left(\frac{dx_0{}^2(t)}{dt^2}\right) + c_1 x_0(t)\left(\frac{dx_0(t)}{dt}\right)^2 + \omega^2 x_1(t) + b_1 x_0(t) = 0, \quad x_1(0) = 0, \quad x_1'(0) = 0 \tag{21}$$

The solution of Eq. (5) is:

$$x_0(t) = \lambda \cos(\omega t) \tag{22}$$

Substituting $x_0(t)$ from Eq. (22) into Eq. (21) results in:

$$\frac{d^2 x_1(t)}{dt^2} - c_1\lambda^3\omega^2 \cos^3(\omega t) + c_1\lambda^3\omega^2 \cos(\omega t)\sin 2(\omega t) + \omega^2 x_1(t) + b_1\lambda\cos(\omega t) = 0, \tag{23}$$

But from Eqs. (17-18) and just with considering the two first terms:

$$b_1 = \frac{A - \omega^2}{p} \tag{24}$$

and

$$c_1 = \frac{4r^2}{p} \tag{25}$$

After setting $p = 1$, eliminating the secular term needs to satisfy the following equation:

$$b_1\lambda - \frac{5}{2}c_1\lambda^3\omega^2 = 0 \tag{26}$$

Two roots of this particular equation can be obtained as:

$$\omega = \pm\sqrt{\frac{A\lambda}{\lambda - 10r^2\lambda^3}} \tag{27}$$

Replacing ω from Eq. (27) into Eq. (22) yields:

$$x(t) = x_0(t) = A\cos\left(\sqrt{\frac{A\lambda}{\lambda - 10r^2\lambda^3}}\,t\right) \tag{28}$$

5. Case 2

5. 1. The rotating rigid frame under force

The rigid frame is forced to rotate at the fixed rate Ω. While the frame rotates, the simple pendulum oscillates (Fig. 2). By using Lagrange method the governing equation can be easily obtained as follows:

$$\frac{d^2 x(t)}{dt^2} + (1 - A\cos(x(t)))\sin(x(t)) = 0 \tag{29}$$

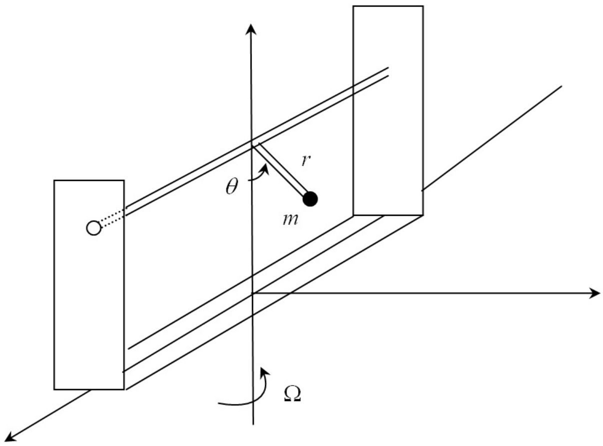

Fig. 2. The geometry of rotating rigid frame under force

Here, by using the Taylor's series expansion for $\cos(x(t))$ and $\sin(x(t))$ the above equation reduces to:

$$\frac{d^2x(t)}{dt^2}+(1-A)x(t)-(\frac{1}{6}x^3(t)+\frac{2}{3}Ax^3(t)-\frac{1}{12}Ax^5(t))=0 \tag{30}$$

With the following boundary conditions:

$$x(0)=\lambda, \qquad x'(0)=0, \tag{31}$$

where A is $\frac{\Omega^2 r}{g}$.

5. 2. Application of HPEM

According to the HPEM, Eq. (30) can be rewritten as:

$$1\frac{d^2x(t)}{dt^2}+(1-A)x(t)-1(\frac{1}{6}x^3(t)+\frac{2}{3}Ax^3(t)-\frac{1}{2}Ax^5(t))=0 \tag{32}$$

and the initial conditions are as follows:

$$x_0(0)=\lambda, \qquad x_0'(0)=0. \tag{33}$$

The form of solution and the constant one in Eq. (32) can be expanded as:

$$x(t)=x_0(t)+p\,x_1(t)+p^2x_2(t)+\dots \tag{34}$$

$$1=1+pa_1+pa_2+\dots \tag{35}$$

$$1-A=\omega^2+pb_1+pb_2+\dots \tag{36}$$

$$1=pc_1+pc_2+\dots \tag{37}$$

Substituting Eqs. (34), through (37) into Eq. (32), and processing as the standard perturbation method, governing equations can be written as:

$$x_0''(t)+\omega^2x_0(t)=0, \qquad x_0(0)=\lambda, \qquad x_0'(0)=0 \tag{38}$$

$$\frac{d^2x_1(t)}{dt^2}+b_1x_0{}^2(t)(\frac{dx_0{}^2(t)}{dt^2})-\frac{1}{2}Ax_0{}^5(t)+\frac{2}{3}Ax_0{}^3(t)+\omega^2x_1(t)+$$
$$+\frac{1}{6}x_0{}^3(t)=0,\ x_1(0)=0,\ x_1'(0)=0 \tag{39}$$

The solution of Eq. (38) is:

$$x_0(t) = \lambda \cos(\omega t) \tag{40}$$

Substituting $x_0(t)$ from the Eq. (40) into Eq. (39) results in:

$$\frac{d^2 x_1(t)}{dt^2} + \frac{1}{6}\lambda^3 \cos^3(\omega t) b_1 \lambda \cos(\omega t) + \frac{2}{3} A\lambda^3 \cos^3(\omega t) + \omega^2 x_1(t) - \\ -\frac{1}{2} A\lambda^5 \cos^5(\omega t) = 0. \tag{41}$$

However from Eqs. (36-37):

$$b_1 = \frac{1 - A - \omega^2}{p} \tag{42}$$

and

$$c_1 = \frac{1}{p} \tag{43}$$

Based on trigonometric functions properties we have:

$$\cos^3(\omega t) = 1/4\cos(3\omega t) + 3/4\cos(\omega t) \tag{44}$$

$$\cos^5(\omega t) = 1/6[\cos(5\omega t) - 5\cos(3\omega t) + 20\cos(\omega t)] \tag{45}$$

After $p = 1$ and eliminating the secular term, ω is obtained as follows:

$$\omega = \pm\sqrt{\frac{1}{8}\lambda^2 + \frac{5}{18}A\lambda^4 - \frac{1}{2}A\lambda^2 + 1 - A} \tag{46}$$

Replacing ω from Eq. (46) into Eq. (40) yields:

$$x(t) = x_0(t) = A\cos\left(\sqrt{\frac{1}{8}\lambda^2 + \frac{5}{18}A\lambda^4 - \frac{1}{2}A\lambda^2 + 1 - A}\, t\right) \tag{47}$$

6. Numerical results

The usefulness of the presented parameter expanding method is investigated by considering two nonlinear dynamic problems explained in previous sections. To validate the HPEM results, convergence studies are carried out and the results are compared with those obtained using numerical results. Figs. 3 and 4 demonstrate the effects of constant parameters on position and velocity profiles versus time, respectively. It is observed that upon application of only one term in series expansion, accurate and reliable solutions can be obtained with validity within the whole solution domain.

7. Conclusions

In this study, analytical solutions for nonlinear problems in dynamics are investigated using hybrid Lagrange and HPEM method as well as another new technique referred to as He's parameter expanding method. Some remarkable advantages and drawbacks of the methods are discussed in more details. Applications of these methods to calculate higher-order approximate periodic solutions to the nonlinear problems are demonstrated. Although HPEM is simple to understand and implement, it is demonstrated that one term in series expansions is sufficient to obtain a highly accurate solutions, which are valid for the whole solution domain. In addition, the ability of the proposed method to predict the response and stability of a dynamical system is shown. The obtained analytical results are in good agreement with those obtained using numerical method. It is observed that the method is a promising tool to solve these types of nonlinear problems with many engineering applications.

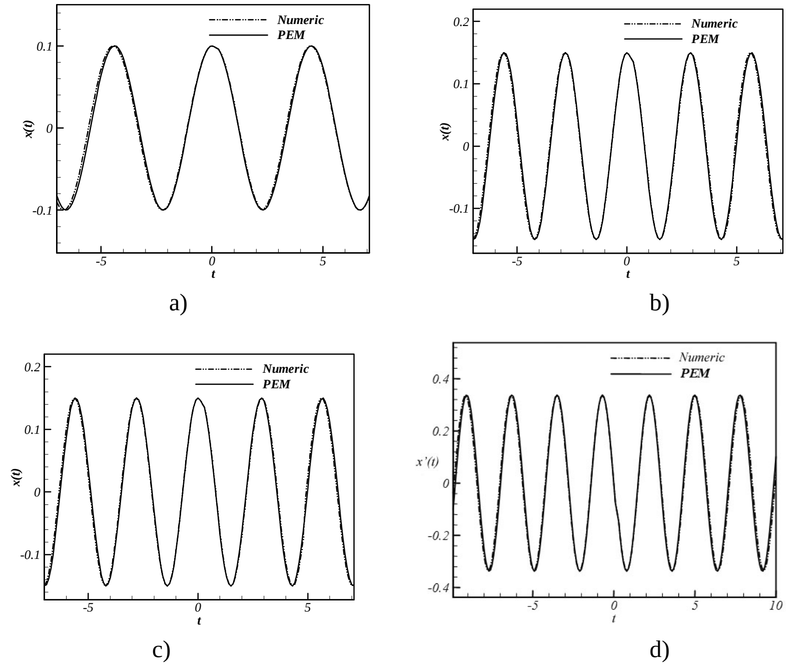

Fig. 3. The effects of constant parameters on position and velocity, example **(1), (**a**)** $A = 2, \lambda = 0.1, r = 0.5$ **,** **(**b**)** $A = 5, \lambda = 0.15, r = 0.25$ **,** **(**c**)** $A = 10, \lambda = 0.1, r = 0.5$ **,** **(**d**)** $A = 5, \lambda = 0.15, r = 0.25$

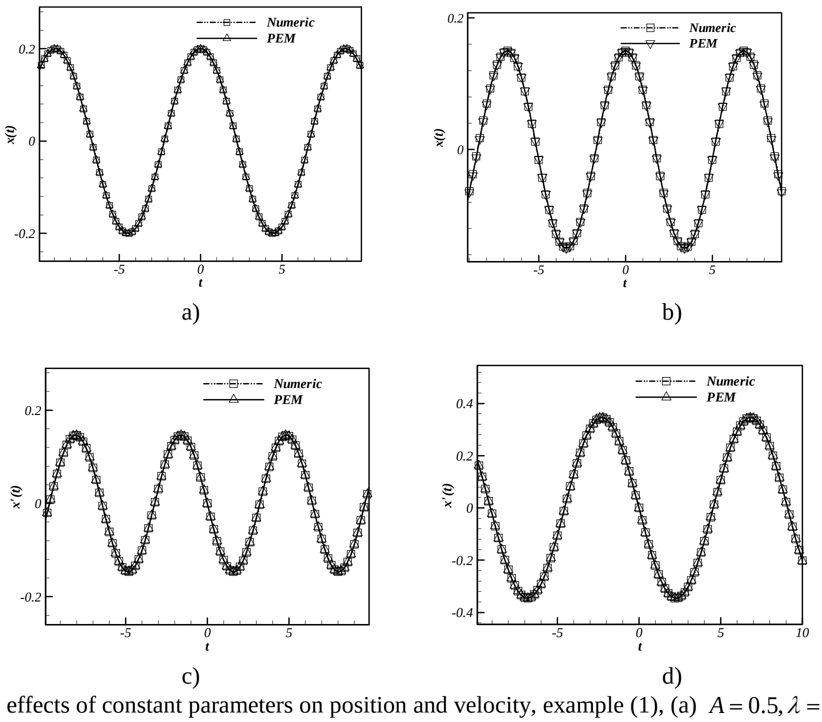

Fig. 4. The effects of constant parameters on position and velocity, example (1), (a) $A = 0.5, \lambda = 0.2$, (b) $A = 0.15, \lambda = 0.15$, (c) $A = 0.05, \lambda = 0.15$, (d) $A = 0.5, \lambda = 0.5$

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