762. On the approximate analytical solution for parametrically excited nonlinear oscillators

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Abstract. In this study we have analytically analyzed the vibration of parametrically excited oscillators based on Mathieu-Duffing equation. He's Variational iteration method (VIM) is applied to obtain analytical solution, while Runge-Kutta method is used to obtain the numerical solution. It is demonstrated that VIM is very effective and convenient therefore may find wide applicability in engineering and other sciences. Finally, to confirm the validity of the applied method, the results of VIM are compared with those obtained by Runge-Kutta method. The results from VIM indicate an excellent agreement with the numerical solutions.

Keywords: Variational iteration method, Runge-Kutta method, parametrically excited oscillator.

Introduction

Many physical phenomena can be classified into linear or nonlinear according to the type of differential equations of motion. Parametrically excited systems are widely spread in many branches of physics and engineering. One of the most important scientific areas is their dynamic behavior. We have difficulty in finding an exact solution for these nonlinear problems and they have to be solved with other approximate analytical methods. Perturbation technique is one of the well-known analytical methods. They are not practical for strongly nonlinear equations, so to conquer the imperfections, novel techniques have appeared in open literature, for instance: Homotopy perturbation [1-2], Energy balance [3-6], Variational approach [7-8], Iteration perturbation method [9] , Max-min approach [10] and other analytical and numerical methods [11-19]. Among these methods, variational iteration method is considered for analysis of the vibration of parametrically excited oscillator based on Mathieu-Duffing equation. The paper has been organized as follows:

In section 1 we describe the basic idea of Variational iteration method (VIM). The basic concept of the Runge-Kutta method is considered in section 2. Then in section 3, applications of He's Variational iteration method (VIM) have been studied to demonstrate the applicability and preciseness of the method for two examples. In section 4, some comparisons between analytical and numerical solutions are presented. Finally, it is demonstrated that results from VIM have an excellent agreement with the numerical ones.

Basic idea of the Variational iteration method

To illustrate the basic concepts of the new technique, we consider the following general differential equation [2]:

$$Lu + Nu = g(x) \tag{1}$$

where, L is a linear operator, and N a nonlinear operator, $g(x)$ an inhomogeneous or forcing term. According to the variational iteration method, we can construct a correct functional as follows:

$$u_{(n+1)}(t) = u_n(t) + \int_0^t \lambda \left\{ Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau) \right\} d\tau \tag{2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. ~ = 0 n δu.

For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

Basic idea of the Runge-Kutta method

For the numerical approach to verify the analytic solution, the fourth order Runge-Kutta method has been used. This iterative algorithm is written in the form of the following formulae for the second-order differential equation:

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{6}\left(h_1 + 2h_2 + 2h_3 + k_4\right)$$
$$u_{i+1} = u_i + \Delta t\left(\dot{u}_i + \frac{\Delta t}{6}\left(h_1 + h_2 + k_3\right)\right) \tag{3}$$

where, Δt is the increment of the time and h_1, h_2, h_3, and h_4 are determined from the following formulae:

$$h_1 = f\left(\dot{u}, u_i, \dot{u}_i\right)k,$$
$$h_2 = f\left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}\dot{u}_i,\ \dot{u}_i + \frac{\Delta t}{2}h_1\right),$$
$$h_3 = f\left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}\dot{u}_i,\ \frac{1}{4}\Delta t^2 h_1,\ \dot{u}_i + \frac{\Delta t}{2}h_2\right), \tag{4}$$
$$h_4 = f\left(t_i + \Delta t,\ u_i + \Delta t\,\dot{u}_i,\ \frac{1}{2}\Delta t^2 h_2,\ \dot{u}_i + \Delta t\, h_3\right).$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative are determined from initial condition. Then, with a small time increment Δt, the displacement function and its first-order derivative at the new position can be obtained using Eq. (3). This process continues to the end of the time limit.

Applications

In order to assess the merits and the accuracy of He's variational iteration method for solving parametrically nonlinear excited oscillator, we will consider the following two examples.

Example 1

The governing equation of Mathieu-Duffing system, which is considered in this study, is described by the following high-order nonlinear differential equation [20]:

$$\ddot{u} + [\delta + 2\varepsilon\ cos(2t)]u - \phi u^3 = 0 \tag{5}$$

where dots indicate differentiation with respect to the time, $\varepsilon \ll 1$ is a small parameter, ϕ is the parameter of nonlinearity, and δ is the transient curve and can be defined as [20]:

$$\delta = \phi u_0^2 \left(1 - \frac{2\varepsilon}{2 + \phi u_0^{\ 2}}\right). \tag{6}$$

The initial condition considered in this study is defined by [20]:

$$u(0) = 0.1,\quad \dot{u}(0) = 0 \tag{7}$$

According to the VIM, we can construct the correction functional of (5) as follows:

$$u_{(n+1)}(t) = u_n(t) + \int_0^\tau \lambda\left\{\ddot{u}_n + [\delta + 2\varepsilon\ cos(2\tau)]u_n - \phi u_n^{\ 3}\right\}d\tau \tag{8}$$

where λ is a general Lagrange multiplier.

Making the above correction functional stationary, we can obtain the following stationary conditions:

$$\lambda''(\tau)=0, \quad \lambda(\tau)_{\tau=t}=0, \quad 1-\lambda'(\tau)\,\big|_{\tau=t}=0. \tag{9}$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda=\tau-t \tag{10}$$

leading to the following iteration formula:

$$u_{(n+1)}(t)=u_n(t)+\int_0^t(\tau-t)\left\{\ddot{u}_n+\left[\delta+2\varepsilon cos\,(2t)\right]u_n-\phi u_n{}^3\right\}d\tau \tag{11}$$

If, for example, the initial conditions are $u(0)=0.1$ and $\dot{u}(0)=0$, we begin with $u_0(t)=0.1$, by the above iteration formula (8) we have the following approximate solutions:

$$u_1(t)=0.1-0.05\varepsilon-0.05\delta t^2+0.05\varepsilon cos\,(2t)+0.0005\phi t^2 \tag{12}$$

In the same way, we obtain as $u_2(t)$ follows:

$$\begin{aligned}
u_2(t)&=0.1-0.05\varepsilon-0.05\delta t^2+0.05\varepsilon cos(2t)+0.0005\phi t^2+0.1875\varepsilon^2\\
&-0.328125\times10^{-3}\phi\varepsilon^2+0.2724609375\times10^{-5}\phi^2\varepsilon^2-0.34722222\times10^{-5}\phi\varepsilon^3 cos^3(2t)\\
&-0.5625\times10^{-5}\varepsilon\phi^2+0.9461805556\times10^{-4}\varepsilon\phi^3+4.6875\times10^{-7}\varepsilon^2\phi^2t^4\\
&+6.696428\times10^{-8}\delta^2\phi^2t^8+1.171875\times10^{-7}\varepsilon^2\phi^2t^2+0.125\times10^{-5}\phi^2t^4\\
&+3.75\times10^{-8}t^3\varepsilon\phi^3 sin(2t)+0.140625\times10^{-4}\varepsilon\delta\phi^2 cos(2t)\\
&+8.4375\times10^{-8}t^2\varepsilon\phi^3 cos(2t)+0.1875\times10^{-5}t^2\varepsilon^2\phi^2 cos(2t)-0.375\times10^{-5}t\varepsilon^2\phi^2 sin(2t)\\
&+0.00025t^2\phi\varepsilon cos(2t)-0.375\times10^{-5}\varepsilon\phi^2 cos(2t)t^2-2.34375\times10^{-7}\varepsilon^2\phi^2 cos^2(2t)t^2\\
&-0.87890625\times10^{-5}\phi\varepsilon^2\delta cos(2t)^2-0.025t^2\delta\varepsilon cos(2t)+0.00028125\phi\varepsilon^2\delta cos(2t)\\
&-0.000703125\phi\varepsilon\delta^2 cos(2t)-0.0005625\phi\varepsilon\delta cos(2t)-0.140625\times10^{-4}\varepsilon\delta\phi^2\\
&+0.05t\,\delta\varepsilon sin(2t)-0.5\times10^{-3}t\,\phi\varepsilon sin(2t)+0.75\times10^{-5}\varepsilon\phi^2 sin(2t)t\\
&-1.125\times10^{-7}t\,\varepsilon\phi^3 sin(2t)-9.375\times10^{-9}t^4\varepsilon\phi^3 cos(2t)+0.5625\times10^{-3}\phi\varepsilon\delta\\
&+0.70312\times10^{-3}\phi\varepsilon\delta^2-0.2724609375\times10^{-3}\phi\varepsilon^2\delta-0.05\delta\varepsilon+0.00075\phi\varepsilon\\
&+7.03125\times10^{-8}\varepsilon\phi^3+0.9461805556\times10^{-4}\phi\varepsilon^3-0.5625\times10^{-5}\varepsilon\phi^2\\
&-0.46875\times10^{-4}\phi\varepsilon^2\delta t^4+0.000125\phi\varepsilon\delta t^4-0.125\times10^{-4}t^6\phi\delta^2\varepsilon\\
&+2.5\times10^{-9}\phi^3t^6+0.2724609375\times10^{-5}\varepsilon^2\phi^2-0.328125\times10^{-3}\phi\varepsilon^2\\
&+0.1875\times10^{-5}t^4\delta\varepsilon\phi^2 cos(2t)+2.23214285710^{-12}\phi^4t^8-7.03125\times10^{-8}\varepsilon\phi^3 cos(2t)\\
&-0.28125\times10^{-5}\varepsilon^2\phi^2 cos(2t)-0.75\times10^{-3}\phi\varepsilon cos(2t)+0.375\times10^{-3}\phi\varepsilon^2 cos(2t)\\
&-0.46875\times10^{-4}\phi\varepsilon^2 cos^2(2t)+0.5625\times10^{-5}\varepsilon\phi^2 cos(2t)+\ldots
\end{aligned} \tag{13}$$

And so on. In the same manner, the rest of the components of the iteration formula can be obtained.

Example 2

For the second parametrically excited nonlinear oscillator we consider the following:

$$\ddot{\theta}+\left[\delta+\varepsilon\cos(t)\right]\sin\theta=0 \tag{14}$$

with the initial conditions:

$$\theta(0)=0.1\,,\dot{\theta}(0)=0 \tag{15}$$

The approximation $sin(\theta)\approx\theta+\frac{1}{6}\theta^3+\frac{1}{120}\theta^5$ is used.

According to the VIM, we can construct the correction functional of (14) as follows:

$$\theta_{(n+1)}(t)=\theta_n(t)+\int_0^t\lambda\left\{\ddot{\theta}_n+\left(\delta+\varepsilon cos(\tau)\right)\left(\theta_n+\frac{1}{6}\theta_n{}^3+\frac{1}{120}\theta_n{}^5\right)\right\}d\tau \tag{16}$$

where λ is general Lagrange multiplier.

Making the above correction functional stationary, we can obtain the following stationary condition:

$$\lambda''(\tau)=0,\quad \lambda(\tau)_{\tau=t}=0,\quad 1-\lambda'(\tau)\,\big|_{\tau=t}=0 \tag{17}$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda=\tau-t \tag{18}$$

Substituting Eq. (18) into the correction functional Eq. (16) results in the following iteration formula:

$$\theta_{(n+1)}(t)=\theta_n(t)+\int_0^t(\tau-t)\left\{\ddot{\theta}_n+\left(\delta+\varepsilon cos(\tau)\right)\left(\theta_n+\frac{1}{6}\theta_n{}^3+\frac{1}{120}\theta_n{}^5\right)\right\}d\tau \tag{19}$$

Now we are to start with an arbitrary initial approximation that satisfies the initial condition:

$$\theta_0(t)=0.1 \tag{20}$$

Substituting (20) into Eq. (19) and after simplifications we have:

$$\theta_1(t)=0.1-0.0998334\varepsilon-0.0499167\delta t^2+0.0998334\varepsilon cos(t) \tag{21}$$

In the same way, we obtain $\theta_2(t)$ as follows:

$$\begin{aligned}
\theta_2(t)&=0.1-0.0998334\varepsilon-0.0499167\delta t^2+0.0998334\varepsilon cos(t)\\
&+0.07450099916\varepsilon^2+0.0006357008267\varepsilon^3-0.1791715023\times10^{-5}\varepsilon^5\\
&-0.02483366639\varepsilon^2t^2+7.243201131\times10\varepsilon^5t^2+1.293428773\times10^{-7}\delta\varepsilon^4t^4\\
&+6.696260722\times10^{-7}\varepsilon^6+1.635149462\times10^{-7}\delta^2\varepsilon^3cos^3(t)-1.84467733810^{-9}\delta^3\varepsilon^3t^8\\
&-4.519459478\times10^{-8}\delta\varepsilon^5cos^4(t)+2.076445331\times10^{-7}\delta\varepsilon^5cos^4(t)\\
&+5.289058863\times10^{-9}\delta\varepsilon^5cos^5(t)-5.165096546\times10^{-8}\delta\varepsilon^5tcos^3(t)sin(t)\\
&-0.3530324359\times10^{-3}\delta^3\varepsilon^3cos(t)t^2+0.1629720254\times10^{-5}t\delta\varepsilon^4cos(t)sin(t)\\
&-0.1237544708\times10^{-3}\delta\varepsilon^3cos^2(t)t^2-1.147799232\times10^{-8}\delta^3\varepsilon^3cos^3(t)t^6\\
&+5.165096546\times10^{-8}\delta^3\varepsilon^3cos^2(t)t^6+7.231135164\times10^{-7}\varepsilon\delta^5cos(t)t^8\\
&-0.7850946750\times10^{-5}\varepsilon\delta^5sin(t)t^7-0.6941889758\times10^{-4}\varepsilon\delta^5cos(t)t^6\\
&-0.1317099619\times10^{-5}\delta^3\varepsilon^3cos^2(t)t^4+0.7205309682\times10^{-5}\delta^3\varepsilon^3cos^2(t)t^2\\
&+2.295598465\times10^{-9}\varepsilon^6cos^6(t)+0.1241691622\times10^{-5}t^3\delta^2\varepsilon^3cos(t)sin(t)\\
&-0.2072937479\times10^{-4}\delta\varepsilon^2t^4+0.09983341667\varepsilon cos(t)+0.1124586141\varepsilon\delta^5cos(t)\\
&-3.099057928\times10^{-7}\delta^3\varepsilon^3t^5cos(t)sin(t)-8.845227835\times10^{-7}\delta\varepsilon^5cos^2(t)\\
&-0.001980071532t^4\delta^3\varepsilon cos(t)+0.8552685694\times10^{-3}\delta^3\varepsilon^3sin(t)t
\end{aligned} \tag{22}$$

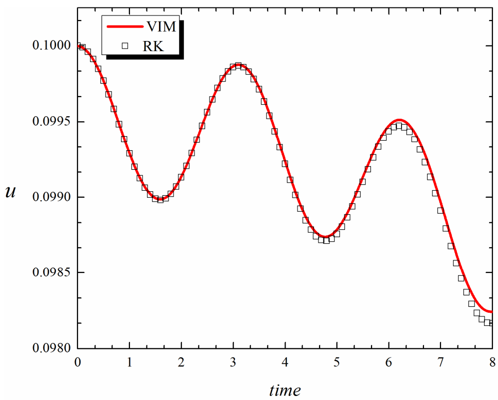
$$+0.2157862557\times10^{-5}\delta^3\varepsilon^3 sin(t)t^5+0.1694151667\times10^{-4}\delta^3\varepsilon^3 cos(t)t^4$$
$$-0.9290542054\times10^{-4}\delta^3\varepsilon^3 sin(t)t^3-3.468904347\times10^{-7}\delta^3\varepsilon^3 cos^3(t)t^2$$
$$+1.377359079\times10^{-7}\delta^3\varepsilon^3 cos^3(t)t^4-0.1876334007\times10^{-4}\delta\varepsilon^4 cos(t)$$
$$-0.2487524974\times10^{-3}\delta\varepsilon^2 tcos(t)sin(t)-0.3725074867\times10^{-5}t^3\delta^3\varepsilon^2 cos(t)sin(t)$$
$$+0.0002359214083\delta^4\varepsilon cos(t)t^4-2.711675687\times10^{-7}\varepsilon^6 t^2$$
$$+5.173715093\times10^{-8}\delta\varepsilon^4 tcos^3(t)sin(t)-0.1124586141\varepsilon\delta^5$$
$$-0.3718869513\times10^{-4}\varepsilon^2\delta^4 sin(t)t^5-0.001013220364\varepsilon^2\delta^3 sin(t)t$$
$$-0.2483383245\times10^{-5}\varepsilon^2\delta^3 sin(t)t^5-0.1986706596\times10^{-4}\varepsilon^2\delta^3 cos(t)t^4$$
$$+0.1092688628\times10^{-3}\varepsilon\delta^3 sin(t)t^3-0.1243762487\times10^{-3}t^4\delta^2\varepsilon cos(t)$$
$$-0.009222796392\varepsilon^2\delta^4 sin(t)t+0.9950099897\times10^{-3}t^3\delta^2\varepsilon sin(t)+...$$

Results and discussion

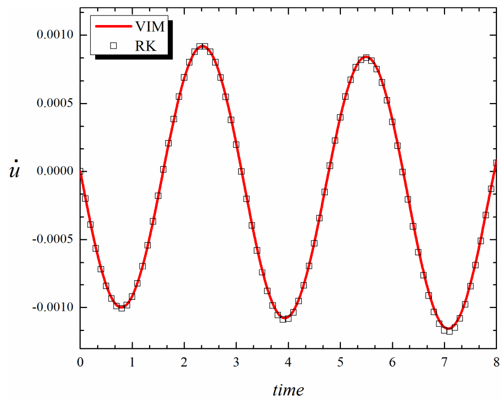
In this study, the Mathieu-Duffing equation has been solved by utilizing VIM method. The results shown in Figures 1–4 indicate that the VIM provides high accuracy. The figures illustrate the time history diagram of the displacement, velocity and phase plan, respectively.

Figures 1a and 1b represent comparison of analytical solution of u and $\dot{u}$ based on time with the numerical solution and Figure 2 shows the comparison of analytical solution of $\dot{u}$ based on u with the numerical solution for example 1.

In example 2, the Figures 3a and 3b indicate that the behavior of the oscillation is periodic. And the comparison of analytical solution of $\dot{\theta}$ based on θ with the numerical solution is shown in Figure 4. In addition, in comparison with Runge-Kutta method, a considerable reduction of the calculation can be observed in the case of VIM. It can be confirmed that VIM is powerful in finding analytical solutions for a wide class of nonlinear problems.



(a)



(b)

Fig. 1. Comparison of analytical solution with the numerical solution for $\phi = 2$, $\varepsilon = 0.01$, $\delta = 0.02$

(a) Time history diagram of u. (b) Time history diagram of $\dot{u}$

Conclusions

The VIM has been applied to analyze the parametrically excited vibrations of oscillators in this study. The results from this method have been compared with those of Runge-Kutta.

Excellent agreement between the two methods is observed. The presented scheme provides concise and straightforward solution to provide reliable results and it overcomes the difficulties associated with the conventional methods. Solution of the Mathieu-Duffing equation indicates that accuracy of the results is considerably affected by the variation of the parameters ε and δ.

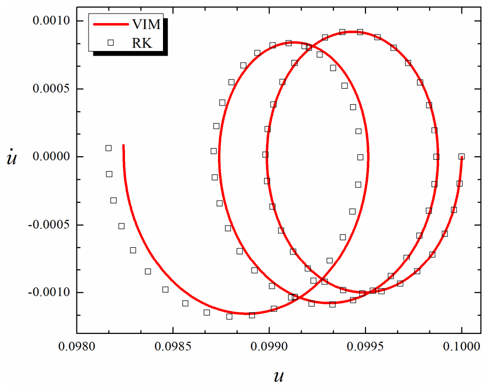
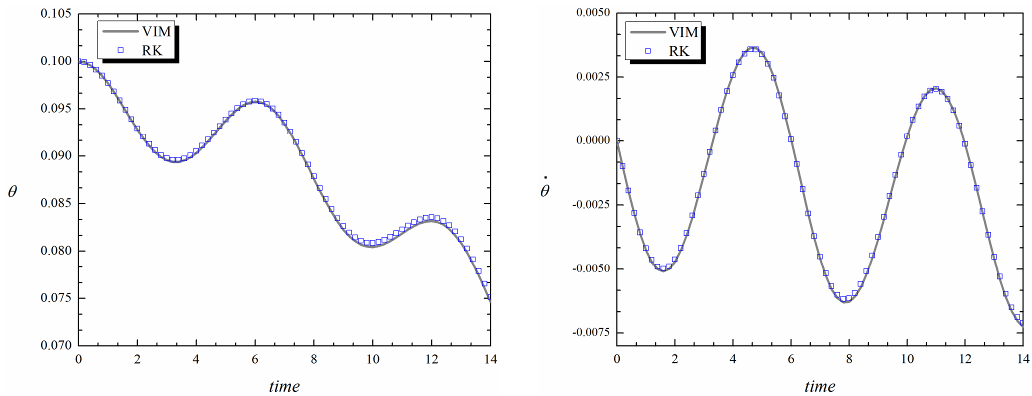


Fig. 2. Comparison of analytical solution with the numerical solution, $\dot{u}$ versus u at ϕ = 2, ε = 0.01, $\delta = 0.02$



(a) (b)

Fig. 3. Comparison of analytical solution with the numerical solution for ε = 0.05, $\delta = 0.001$

(a) Time history diagram of θ . (b) Time history diagram of $\dot{\theta}$

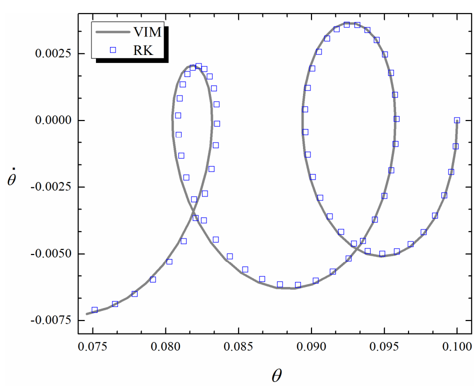


Fig. 4. Comparison of analytical solution with the numerical solution, $\dot{\theta}$ versus θ at ε = 0.05, $\delta = 0.001$

References

[1] **Bayat M., Shahidi M., Barari A., Domairry G.** The approximate analysis of nonlinear behavior of structure under harmonic loading. Int. J. Phy. Sci., Vol. 5, Issue 7, 2010, p. 1074-1080.

[2] **He J. H.** Homotopy perturbation method for bifurcation of nonlinear problems. Int. J. Nonlinear Sci. Numer. Simulat., Vol. 6, Issue 2, 2005, p. 207-208.

[3] **Bayat M., Shahidi M., Barari A., Domairry G.** Analytical evaluation of the nonlinear vibration of coupled oscillator systems. Zeitschrift fur Naturforschung, Section A-A, Journal of Physical Sciences, Vol. 66, Issue 1-2, 2011, p. 67-74.

[4] **Bayat M., Barari A., Shahidi M.** Dynamic response of axially loaded Euler-Bernoulli beams. Mechanika, Vol. 17, Issue 2, 2011, p. 172-177.

[5] **Pakar I., Bayat M.** Analytical solution for strongly nonlinear oscillation systems using Energy Balance Method. Int. J. Phy. Sci., Vol. 6, Issue 22, 2011, p. 5166-5170.

[6] **Bayat M., Pakar I.** Application of He's Energy Balance Method for nonlinear vibration of thin circular sector cylinder. Int. J. Phy. Sci., Vol. 6, Issue 23, 2011, p. 5564-5570.

[7] **Bayat Mahdi, Bayat Massoumeh, Bayat Mahmoud** An analytical approach on a mass grounded by linear and nonlinear springs in series. Int. J. Phy. Sci., Vol. 6, Issue 2, 2011, p. 229-236.

[8] **Shahidi M., Bayat M., Pakar I., Abdollahzadeh G. R.** On the solution of free non-linear vibration of beams. Int. J. Phy. Sci., Vol. 6, Issue 7, 2011, p. 1628-1634.

[9] **Bayat Mahmoud, Shahidi Mehran, Bayat Mahdi** Application of Iteration Perturbation Method for nonlinear oscillators with discontinuities. Int. J. Phy. Sci., Vol. 6, Issue 15, 2011, p. 3608-3612.

[10] **Bayat Mahmoud, Iman Pakar, Bayat Mahdi** Analytical study on the vibration frequencies of tapered beams. Latin American Journal of Solids and Structures, Vol. 8, Issue 2, 2011, p. 149-162.

[11] **Soleimani Kutanaei S., E. Ghasemi, M. Bayat** Mesh-free modeling of two-dimensional heat conduction between eccentric circular cylinders. Int. J. Phy. Sci., Vol. 6, Issue 16, 2011, p. 4044-4052.

[12] **Ghasemi Esmail, Bayat Mahmoud, Bayat Mahdi** Visco-elastic MHD flow of Walters liquid B fluid and heat transfer over a non-isothermal stretching sheet. Int. J. Phy. Sci., Vol. 6, Issue 21, 2011, p. 5022-5039.

[13] **Pakar I., Bayat M., Bayat M.** Analytical evaluation of the nonlinear vibration of a solid circular sector object. Int. J. Phy. Sci., Vol. 6, Issue 30, 2011, p. 6861-6866.

[14] **Bayat M., Pakar I.** Nonlinear free vibration analysis of tapered beams by Hamiltonian approach. Journal of Vibroengineering, Vol. 13, Issue 4, 2011, p. 654-661.

[15] **Bayat M., Pakar I., Shahidi M.** Analysis of nonlinear vibration of coupled systems with cubic nonlinearity. Mechanika, Vol. 17, Issue 6, 2011, p. 620-629.

[16] **Bayat M., Abdollahzadeh G. R.** Analysis of the steel braced frames equipped with ADAS devices under the far field records. Latin American Journal of Solids and Structures, Vol. 8, Issue 2, 2011, p. 163-181.

[17] **Bayat M., Abdollahzadeh G. R.** On the effect of the near field records on the steel braced frames equipped with energy dissipating devices. Latin American Journal of Solids and Structures, Vol. 8, Issue 4, 2011, p. 429-443.

[18] **He J. H.** Variational iteration method: a kind of nonlinear analytical technique: some examples. International Journal of Non-Linear Mechanics, Vol. 34, Issue 4, 1999, p. 699-708.

[19] **He J. H.** Some new approaches to Duffing equation with strongly and high order nonlinearity (II) Parameterized perturbation technique. Communications in Nonlinear Science and Numerical Simulation, Vol. 4, Issue 1, 1999, p. 81-83.

[20] **Andrianov I. V., Awrejcewicz J., Manevitch L. I.** Asymptotical Mechanics of Thin-Walled Structures. Germany, Springer-Verlag, Berlin, Heidelberg, 2004.