853. Analytical model to determine fundamental frequency of free vibration of perforated plate by using unit step functions to express non-homogeneity

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Abstract. In the current study an analytical model to determine fundamental frequency of perforated plate is formulated. Non-homogeneity in Young's modulus and density due to perforation is expressed by using unit step function in Rayleigh's Quotient. In the present analysis the boundary condition considered is clamped at all edges. Perforated plate is considered as plate with uniformly distributed mass and holes are considered as non-homogeneous patches. The deflected middle surface of the plate is approximated by a function which satisfies the boundary conditions. The proposed approach is validated by comparing results with finite element method modal analysis.

Keywords: perforated plate, non-homogeneous plate, unit step function, Rayleigh's method, vibration of plate.

1. Introduction

Perforated plates are widely used in nuclear power equipment, heat exchangers and pressure vessels. The holes in the plate are arranged in various regular penetration patterns. Industrial applications include both square and triangular array perforation patterns. Cutouts are found in mechanical, civil, marine and aerospace structures commonly as access ports for mechanical and electrical systems, or simply to reduce weight. Cutouts are also made to provide ventilation as well as to modify the resonant frequency of structures.

Leissa [1-7] compiled some works done in the field of homogeneous and non-homogeneous plates. Sobczyk [8] studied free transverse vibrations of elastic rectangular plates with random material properties. Laura et al. [9] studied non-homogeneous rectangular plates. Mishra and Das [10] proposed a method of characteristic orthogonal polynomials in one dimension to handle rectangular plates. Pan [11] developed characteristic orthogonal polynomials in two variables to study flexural vibrations of polygonal plates. Rao et al. [12-13] studied vibrations of inhomogeneous thin plates using a high precision triangular element, and vibration of inhomogeneous rectangular plates by using perturbation solution. Tomar et al. [14-17] studied vibrations of plates of variable thickness having non-homogeneity. Chakraverty et al. [18-19] used two-dimensional orthogonal polynomials as shape functions in the Rayleigh-Ritz method to study vibration of non-homogeneous plates; they also studied effect of non-homogeneity on natural frequencies of vibration of elliptic plates. Lal R. et al. [20] studied free transverse vibrations of uniform non-homogeneous rectangular plates using boundary characteristic orthogonal polynomials in the Rayleigh-Ritz method on the basis of classical plate theory for four different combinations of clamped, simply supported and free edges. Lal R. and Dhanpati [21] studied effect of non-homogeneity on vibration of orthotropic rectangular plates of varying thickness resting on Pasternak foundation.

From the literature of perforated plate authors found that papers [22-24] deal with equivalent properties of material for perforated plate. These equivalent material properties are used in vibration analysis to consider perforated plate as full solid plate. Burgemeister K. A. and Hansen C. H. [25] showed that to predict accurately the resonance frequencies of simply supported perforated panel, effective material constants cannot be used in classical equations. They used cubic function fitted from ANSYS results to determine the effective resonance frequency ratio for large range of panel geometries with an error of less than 3 %. Mali K. D. and Singru P. M. [26] used Galerkin method for determining the fundamental frequency of rectangular perforated plate with rectangular perforation pattern of circular holes. Perforated plate was considered as a plate with uniformly distributed mass and holes were considered as concentrated negative masses. Mali K. D. and Singru P. M. [27] formulated an analytical model to determine fundamental frequency of free vibration of perforated plate by using greatest integer functions to express non-homogeneity. From review of the literature, authors have found no work dealing with analytical formulation by considering unit step function to express non-homogeneity due to holes (in Young's modulus and density). Present approach can be used to predict accurately the fundamental frequencies of wide range of perforation geometries, for rectangular plates with rectangular penetration pattern for different support condition.

In this paper, Rayleigh's formulation for perforated plate of uniform thickness is carried out by considering non-homogeneity in material properties i.e. Young's modulus and density. In this work perforation pattern considered is square with square perforations having ligament efficiency of 0.5. Boundary condition considered is clamped at all edges. To validate the accuracy of the mathematical model, finite element method (FEM) analysis results for different plates are presented.

2. Analytical formulation

Fundamental frequency expression of a thin plate of uniform thickness is formulated by Rayleigh's principle [1, 28]. Rayleigh's quotient for fundamental frequency of homogeneous thin plate is given by equation:

$$\omega^2 = \frac{\iint_R D_0 \left[\left(\nabla^2 W_1\right)^2 + 2\left(1-\upsilon_0\right)\left\{ \left(\frac{\partial^2 W_1}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 W_1}{\partial x^2}\right)\left(\frac{\partial^2 W_1}{\partial y^2}\right) \right\} \right] dxdy}{\iint_R h\rho_0 W_1^2 \, dxdy} \tag{1}$$

where ω is fundamental frequency, h is the uniform plate thickness, ρ_0 is the density, ν_0 is the Poisson's ratio, W_1 is shape function, and R is the rectangular area over which integration is performed. D_0 is the flexural rigidity, ∇^2 is two-dimensional Laplacian operator. D_0 and ∇^2 are given as:

$$D_0 = \frac{E_0 h^3}{12\left(1-\upsilon_0^2\right)}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2}$$

where E_0 is modulus of elasticity.

The Rayleigh's quotient depends on the form of the function W_1. The function $W_1(x, y)$ is a continuous function that approximately represents the shape of the plate's deflected middle surface and satisfies at least the kinematic boundary conditions and ω represents the natural frequency of the plate pertinent to the assumed shape function. Assume $\omega = \omega_1$ be its fundamental frequency.

2. 1 Fundamental frequency estimation for plates with perforations

In present analysis, square plate with square perforations is considered as shown in Fig. 1. The ligament efficiency ηl, considered for perforation, is $(p/d + p) = 0.5$, where p is ligament length (p_x or p_y) and d is side length of square perforation. Model in the present work does not consider any rotary inertia of the plate. To approximate the shape of the plate's deflected middle surface, function $W_1(x, y)$ used is given as:

$$W_1(x,y)=\left(x^2-\frac{a^2}{4}\right)^2\left(y^2-\frac{b^2}{4}\right)^2 \tag{3}$$

where a and b are side dimensions of the plate along x and y directions respectively.

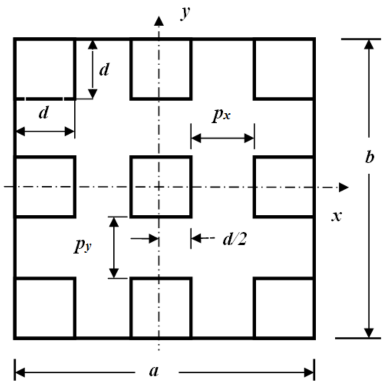

Fig. 1. Coordinates of perforated plate

The shape function $W_1(x, y)$ satisfies the following boundary conditions for the plate clamped on all edges:

$$\left(\frac{\partial W_1(x,y)}{\partial x}\right)_{x=\pm a/2}=0,\ \left(\frac{\partial W_1(x,y)}{\partial y}\right)_{y=\pm b/2}=0,$$
$$W_1(x,y)|_{x=\pm a/2}=0,\ W_1(x,y)|_{y=\pm b/2}=0. \tag{4}$$

From equation (1) the Rayleigh's quotient depends on the material properties like density (ρ_0), modulus of elasticity (E_0), Poisson's ratio (ν_0). For a perforated plate as shown in Fig. 1, the density (ρ), and modulus of elasticity (E) are changing along the surface of the plate with geometric pattern of holes [27]. The pattern of the variation of these parameters along the surface resembles that of square waves. To evaluate the integrals involved in Rayleigh's quotient the density and the modulus of elasticity need to be expressed as a function of the Cartesian co-ordinates x and y. If the function $F(x, y)$ represents the variation of these parameters along the surface, then the density and modulus of elasticity can be expressed as:

$$\rho=\rho_0 F(x,y)$$
$$E=E_0 F(x,y) \tag{5}$$

where E_0 and ρ_0 are the modulus of elasticity and density for a homogeneous plate. Once the function $F(x, y)$ is constructed the integrals can be evaluated using above equations (1-3). The Rayleigh's quotient now becomes [27-28]:

$$\omega_1^2 = \frac{E_0 h^2}{12\rho_0\left(1-\upsilon_0^2\right)} \frac{\iint_R F(x,y)\left[\left(\nabla^2 W_1\right)^2 + 2\left(1-\upsilon\right)\left\{\left(\frac{\partial^2 W_1}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 W_1}{\partial x^2}\right)\left(\frac{\partial^2 W_1}{\partial y^2}\right)\right\}\right] dxdy}{\iint_R F(x,y) W_1^2 dxdy} \tag{6}$$

The function $F(x, y)$ represents the variation of the density and modulus of elasticity. For the function $F(x, y)$ to represent these parameters it must satisfy the following requirements:

$F(x, y) = 0$ in the region corresponding to a perforation,

$F(x, y) = 1$ otherwise.

The function $F(x, y)$ is constructed as per the geometry of the plates considered. To construct the function $F(x, y)$ we assume that density at any point (x, y) is the superposition of the density along x and y directions [27]. This superposition is also considered for modulus of elasticity. The functions $f(x)$ and $g(y)$ represent variation of density and modulus of elasticity along x and y axes respectively. Equations (8) and (9) show the rectangular Heaviside function used to express the non-homogeneity in Young's modulus and density of the plate due to perforations. The functions $f(x)$ and $g(y)$ are formed by using unit step functions and are superimposed to obtain the function $F(x, y)$.

The unit step as a function of a discrete variable n is given as:

$$H(n) = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases} \tag{7}$$

where n is an integer.

$$\begin{aligned} f(x) &= H(x-\frac{d}{2}) - H(x-\frac{d}{2}-p_x) - H(x-\frac{d}{2}-p_x-d) + \\ &+((H(-x-\frac{d}{2}) - H(-x-\frac{d}{2}-p_x) - H(-x-\frac{d}{2}-p_x-d)) \end{aligned} \tag{8}$$

$$\begin{aligned} g(y) &= H(y-\frac{d}{2}) - H(y-\frac{d}{2}-p_y) - H(y-\frac{d}{2}-p_y-d) + \\ &+((H(-y-\frac{d}{2}) - H(-y-\frac{d}{2}-p_y) - H(-y-\frac{d}{2}-p_y-d)) \end{aligned} \tag{9}$$

where d is side length of the square perforation, p_x and p_y are ligament lengths in x and y directions respectively.

In present study square plate having square perforation is considered as shown in Fig. 1. For plate with $\eta l = 0.5$, $p_x = p_y = d$ the above expressions for $f(x)$ and $g(y)$ become:

$$\begin{aligned} f(x) &= H(x-\frac{d}{2}) - H(x-\frac{3d}{2}) - H(x-\frac{5d}{2}) + \\ &+((H(-x-\frac{d}{2}) - H(-x-\frac{3d}{2}) - H(-x-\frac{5d}{2})) \end{aligned} \tag{10}$$

$$g(x) = H(y - \frac{d}{2}) - H(y - \frac{3d}{2}) - H(y - \frac{5d}{2}) + \\ +((H(-y - \frac{d}{2}) - H(-y - \frac{3d}{2}) - H(-y - \frac{5d}{2})) \tag{11}$$

The superposition of $f(x)$ and $g(y)$ to obtain $F(x, y)$ is analogous to Boolean operation of union. Also the functions $f(x)$ and $g(y)$ are independent. $F(x, y)$ can be obtained from the relation:

$$F(x, y) = f(x) + g(y) - f(x)g(y) \tag{12}$$

$F(x, y)$ thus obtained [27] is used in Rayleigh's quotient, equation (6) to obtain the fundamental frequency. These calculations were performed for plates of different sizes.

3. Numerical simulation

Analytical model developed in section 2.1 is applicable to rectangular perforated plates with different side dimensions and having rectangular perforation and, provided that the perforation pattern is rectangular or square and all perforations are of same size. A square plate with simple geometry was considered for convenience of computation. By virtue of the symbolic forms presented in this work, the method can be applied to analytical studies of perforated plates with different boundary conditions. Numerical results have been obtained for the five specimens listed in Table 1 and fundamental natural frequencies are provided in Table 2. The material properties used for all the analyzed specimen plates:
$E_0 = 2.1\times10^{11}$ N/m^2, $\upsilon_0 = 0.3$, $\rho_0 = 7850$ kg/m^3.

Table 1. Specimen parameters

Specimen No.	Plate size (a mm × b mm)	Cutout size (d mm × d mm)	h (mm)	$p_x = p_y$ (mm)
1	400 × 400	80 × 80	2	80
2	500 × 500	100 × 100	2	100
3	600 × 600	120 × 120	2	120
4	700 × 700	140 × 140	2	140
5	800 × 800	160 × 160	2	160

Table 2. Comparison of FEM and analytical results

Specimen No.	Fundamental frequency ω_1, (Hz)		% Error
	Analytical	FEM	
1	102.269	94.411	8.32
2	65.452	60.421	8.32
3	45.452	41.961	8.31
4	33.393	30.828	8.32
5	25.567	23.602	8.32

3. 1. FEM Analysis

To verify the validity of the proposed model, modal analysis is carried out using FEM for clamped steel plates having 2 mm thickness and carrying nine holes at positions shown in Fig. 1. Parameters of the plate specimen considered in this study are listed in Table1. Modal analysis is carried out in ANSYS 11 using Shell63 elements. Meshing is executed by free meshing with smart size option and quadrilateral elements are used. Mesh convergence for FEM results is checked for every specimen. This is checked by running different simulations. Final solution is chosen based on the mesh quality as well as mesh size [26-27]. Thus, the converged solution is the one with the lowest eigenfrequencies given in Table 2. It is assumed that structure is formed of isotropic homogeneous elastic material, i.e. mild steel with material properties same as used in numerical analysis.

4. Results and Discussions

Comparison of natural frequency of the first mode of vibration between the proposed analytical model and FEM is given in Table 2.

The agreement between the analytical approach and the FEM results is reasonably good. It is observed that the difference between numerical and FEM results gives systematic error of 8.32 %. In numerical simulation the mass matrix is formed using same shape function as used to generate stiffness matrix, i.e. the two matrices are consistent one with the other. Thus, numerical simulation predicts higher values of frequencies. This performance of the proposed model, for square/rectangular holes is due to the fact that the effects of both the different holes and their locations on the frequency have been accounted by constructing special function consisting unit step function, to express variation in density and Young's modulus.

Systematic error of 8.32 % occurs between numerical and FEM results because of the following reasons.

1) All the specimens have the same mass remnant ratio ($MRR = 0.64$) i.e ratio of the mass of the perforated plate to the mass of the solid plate of equal outer effective dimensions.

2) Mass remnant ratio depends upon the geometrical parameters such as specimen aspect ratio (a/b), perforation aspect ratio (d/d), thickness (h) of the specimen and ligament efficiency (ηl). All these geometrical parameters are identical for the specimens considered.

3) Due to the geometrical similarity of the specimens the same percentage (%) error occurs between numerical and FEM results for each specimen, though absolute error is different.

This systematic error demonstrates that the analytical model as given, for square plate with square perforations having $\eta l = 0.5$ gives results with the same accuracy for plates with geometrical similarity but variation in dimensions.

5. Conclusion

This work presents an analytical model to estimate the fundamental frequency of thin plates of uniform thickness having square perforations in a square pattern. The effect of non-homogeneity in Young's modulus and density due to holes on the natural frequency of perforated plate has been modeled using unit step functions in the Rayleigh's method. The proposed model has been verified by comparing the numerical results with FEM simulations (Table 2). It was determined that the error in the fundamental frequency is of the order of 8.32 %. Thus, fundamental frequency of the perforated plate can be obtained by a proper choice of various plate parameters and shape function depending on the boundary condition.

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