878. Numerical simulation and analysis of the micro electrostatic device

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(Received 28 June 2012; accepted 4 December 2012)

Abstract. In general, analysis of the electrostatic device is quite difficult and complicated due to the electrostatic coupling effect and the nonlinear electrostatic force. In this study, a hybrid method (H. M.) for the micro-structure system, which combines the differential transformation method (D. T. M.) and finite difference approximation techniques, is used to overcome the nonlinear electrostatic coupling phenomenon. The differential transformation employed is a transformed function based on the Taylor series that is effective in solving nonlinear problems with fast convergence. First, the natural frequencies of a micro fixed-fixed beam are derived as the solutions to a boundary value problem with prescribed boundary conditions by the differential transformation method. And then the nonlinear governing equation of a micro cantilever beam is solved by the hybrid method. The numerical results of the calculated pull-in voltage and natural frequencies are compared with other literatures using various computational methods and are found to be in good agreement. Overall, the results presented in this study show that the proposed hybrid method, provides an accurate and versatile means of analyzing the complex nonlinear behavior of a micro electrostatic devices.

Keywords: micro cantilever beam, micro fixed-fixed beam, electrostatic device, differential transformation.

Introduction

In recent years, many commercial products using microelectromechanical devices have been developed. These have a wide range of uses in many fields; for example, in sensor devices such as accelerometers and pressure sensors [1] for automotive security systems, and in actuators such as the Digital Micromirror Device (DMD) and the electrostatic rotary comb actuator [2]. All of these devices employ electrostatic force to accomplish beam deformation for the purpose of sensing and actuating, hence the electrostatic actuation is the most popular for use in micro-structure systems. Actually, in the electrostatic actuation of a micro-structure system, the electrostatic force is produced from the voltages of two electrodes. If the electrostatic force is greater than the elastic restoring force of the micro-structure system, this represents an unstable phenomenon [3], and the two electrodes attract and come into contact with each other suddenly. The critical value of the voltage is defined as the pull-in voltage, which has a tremendous influence on the electrostatic device. For example, the electrostatic device is regarded as an actuator when the operation voltage is greater than the pull-in voltage and the upper electrode can be attracted to the fixed bottom electrode very quickly; therefore, the pull-in voltage limits the operation range of the actuator. The pull-in behavior phenomenon, however, can be used in the design of such components as switches [4] and relays; moreover, it could be used to measure Young's modulus and residual stress values [3]. Hence, the pull-in voltage is a very important parameter in the design of microelectromechanical devices. Hung et al. examined the leveraged bending and strain-stiffening methods for extending the travel distance before the occurrence of electrostatic actuator pull-in [5]. Chan et al. measured the pull-in voltage and capacitance-voltage together with two-dimensional simulations to extract material properties [6]. Nemirovsky et al. presented a generalized model of pull-in voltage [7]. Zook and Burns [8] proposed two finite element models, namely beam and plate models that consider the shear deformation and rotary inertia, but do not include midplane stretching, to calculate the natural frequencies of a micro-beam subject to an axial load. Tilmans [9] used Rayleigh's method to calculate the natural frequencies of the clamped-clamped micro-beam.

Zhao proposed the differential transformation theory in 1986, which is applied to solve the linear and nonlinear initial value problems of circuit analysis. Recently, researchers have used this theory to solve initial value problems in mechanical engineering. Chen et al. first applied the differential transformation theory to solve the eigenvalue problem [10-11]. Chen et al. [12] showed that the combined differential transformation and finite difference method provides a precise and computationally-efficient means of analyzing the nonlinear dynamic behavior of fixed-fixed micro-beams. The same group also used the hybrid method to analyze the nonlinear dynamic response of an electrostatically-actuated micro circular plate subject to both effects of residual stress and a uniform hydrostatic pressure on the upper surface [13-14]. A numerical investigation was performed into the entropy generated within a mixed convection flow with viscous dissipation effects in a parallel-plate vertical channel using differential transformation method by Chen et al. [15].

The study begins with an explanation of the differential transformation theory, and then the differential transformation method is employed to the governing equation of a micro fixed-fixed beam. The hybrid method is used to complete the nonlinear partial differential equation of the micro cantilever beam and specify the initial conditions and boundary conditions. Finally, numerical results are derived using the hybrid method and are compared with other literature results.

Differential Transformation Theory

The basic principles of the differential transformation method are introduced below.

If $x(t)$ is an analyzable function in time domain T, a definition of the differential transformation of x at $t = t_0$ in the K domain is:

$$X(k;t_0) = M(k)\left(\frac{d^k}{dt^k}(q(t)x(t))\right)_{t=t_0}, \quad k \in K \tag{1}$$

where k belongs to the set of non-negative integers denoted as the K domain, $X(k;t_0)$ is the transformed function in the transformation domain, otherwise called the spectrum of $x(t)$ at $t = t_0$ in the K domain, $M(k)$ is the weighting factor, and $q(t)$ is regarded as a kernel corresponding to $x(t)$. Both $M(k)$ and $q(t)$ are non-zero and $q(t)$ is an analyzable function in time domain T. Therefore, the differential inverse transformation of $X(k;t_0)$ can be described as:

$$x(t) = \frac{1}{q(t)}\sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\frac{X(k;t_0)}{M(k)}, \qquad t \in T \tag{2}$$

if $M(k) = H^k/k!$ and $q(t) = 1$, where H is the time interval. Let $t_0 = 0$; Eq. (1) then becomes:

$$X(k) = \frac{H^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}, \qquad k \in K \tag{3}$$

The differential inverse transformation of $X(k)$ can then be expressed as below by Eq. (2):

$$x(t)=\sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^{k}X(k),\ t\in T \tag{4}$$

Substituting Eq. (3) into Eq. (4) gives:

$$x(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0},\ t\in T \tag{5}$$

Eq. (5) can be derived by Taylor series expansion. Therefore, the main basic operation properties of the differential transform are as listed below:

(a) Linearity operation:

$$T[\alpha x(t)+\beta y(t)]=\alpha X(k)+\beta Y(k) \tag{6}$$

where *T* denotes the differential transform and α and β are any real numbers.

(b) Differential operation:

$$T\left[\frac{d^n x(t)}{dt^n}\right]=\frac{(k+n)!}{k!H^n}X(k+n) \tag{7}$$

where *T* denotes the differential transform and *n* is the order of differentiation [12–15].

Natural Frequencies Analysis of the Micro Fixed-Fixed Beam

The purpose of this section is to obtain the governing equation of the system using Hamilton's principle. First, using energy expressions that include the strain energy, electrical potential energy and kinetic energy, Hamilton's principle is used to obtain the partial differential equation [14]. The micro fixed-fixed beam is shown schematically in Fig. 1. The governing equation of micro fixed-fixed beam as follows:

$$\tilde{E}I\frac{\partial^4 u}{\partial x^4}+\rho A\frac{\partial^2 u}{\partial t^2}=\frac{\varepsilon_0 w V^2}{2(G-u)^2} \tag{8}$$

where ε_0, w, V and G represent the permittivity of free space, the width of the micro fixed-fixed beam, the voltage applied between electrodes and the initial gap between electrodes, respectively. ρ is the density of micro fixed-fixed beam and *A* is the cross-section of micro fixed-fixed beam $\tilde{E}$ is the effective beam material modulus. The micro fixed-fixed beam is considered wide when $w\geq 5h$ that $\tilde{E}$ for wide beam becomes the plate modulus $E/(1-v^2)$, where *v* is the Poisson ratio and *h* is the thickness of the beam, or else for the narrow beam which $\tilde{E}$ is equal to Young's modulus, *E*. *I* is the moment of inertia of the micro fixed-fixed beam. The transverse displacement *u* is the function of position *x* and time *t* and can be shown as $u=u(x,t)$.

The boundary condition is, for example:

$$u(x,t)=\frac{\partial u(x,t)}{\partial x}=0 \quad \text{at} \quad x=0$$
$$u(x,t)=\frac{\partial u(x,t)}{\partial x}=0 \quad \text{at} \quad x=L \tag{9}$$

and the initial condition is, for example:

$$u(x,0)=\frac{\partial u(x,0)}{\partial t}=0 \tag{10}$$

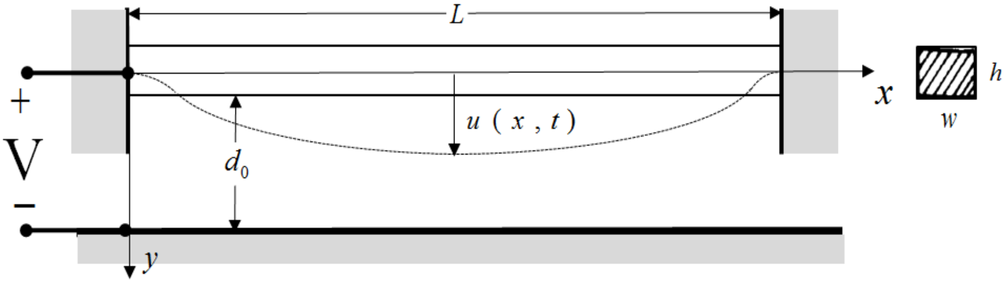

Fig. 1. A schematic diagram of the micro fixed-fixed beam

Not taking into account the effects of residual stress and stray electric field, the governing equation, in dealing with a free vibrating micro fixed-fixed beam of length L, as illustrated in Fig. 1, is simplified into:

$$\tilde{E}I\frac{\partial^4 u}{\partial x^4}+\rho A\frac{\partial^2 u}{\partial t^2}=0 \tag{11}$$

For an arbitrary vibration mode, the transverse displacement u can be expressed as:

$$u(x,t)=\bar{U}(x)T(t) \tag{12}$$

where $\bar{U}(x)$ represents the modal displacement of the micro fixed-fixed beam, and $T(t)$ a time harmonic function. With ω symbolizing the angular frequency, then:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=-\omega^2\bar{U}(x) \tag{13}$$

Consequently, the eigenvalue problem in Eq. (13) is simplified into the ordinary differential equation:

$$\tilde{E}I\frac{\partial^4 \bar{U}(x)}{\partial x^4}-\rho A\omega^2\bar{U}(x)=0 \tag{14}$$

For brevity, dimensionless quantities are defined as:

$$\bar{x} = \frac{x}{L}, \ \ \phi(\bar{x}) = \frac{\bar{U}(x)}{d_0}, \ \ \bar{\Omega}^2 = \frac{\rho A \omega^2 L^4}{\tilde{E}I}. \tag{15}$$

Respective substitutions of Eq. (15) into Eqs. (13) and (14) yield:

$$\frac{\partial^4 \phi(\bar{x})}{\partial \bar{x}^4} - \bar{\Omega}^2 \phi(\bar{x}) = 0 \tag{16}$$

and the dimensionless boundary condition is:

$$\phi(\bar{x}) = \frac{d\phi(\bar{x})}{\partial \bar{x}} = 0 \quad \text{at} \quad \bar{x} = 0$$
$$\phi(\bar{x}) = \frac{d\phi(\bar{x})}{\partial \bar{x}} = 0 \quad \text{at} \quad \bar{x} = 1 \tag{17}$$

It is devoted to firstly address eigenfunctions by use of a differential transformation approach, and secondly natural frequencies by mode superposition. Letting $\lambda = \bar{\Omega}^2$, taking the differential transformation of Eq. (16) leads to:

$$\sum_{r=0}^{k} (r+1)(r+2)(r+3)(r+4)\Phi(r+4) = \sum_{r=0}^{k} \lambda \Phi(r) \, , \tag{18}$$

with boundary conditions, for $\bar{x} = 0$:

$$\Phi(0) = 0, \quad \Phi(1) = 0, \tag{19}$$

and for $\bar{x} = 1$:

$$\sum_{k=0}^{m} \Phi(k) = 0, \quad \sum_{k=0}^{m} k\Phi(k) = 0. \tag{20}$$

$\bar{\Omega}_i^{(k)}$ and $\bar{\Omega}_i^{(k-1)}$ denote the respective ith dimensionless natural frequencies through a kth order differential transformation and a $(k-1)$th one, while the order is determined by the condition:

$$\left| \bar{\Omega}_i^{(k)} - \bar{\Omega}_i^{(k-1)} \right| \le \varepsilon \, , \tag{21}$$

for a specified error $\varepsilon > 0$. Accordingly, in the case of $k = 33$, it is found that:

$$\bar{\Omega}_1^{(33)} = 22.3722, \quad \bar{\Omega}_2^{(33)} = 61.6728. \tag{22}$$

The inequality (21) holds true for $\bar{\Omega}_1 = 22.3722$, $\bar{\Omega}_2 = 61.6728$ and ε = 0.0001. Through mode superposition, the first, second and third vibration modes are plotted in Fig. 2.

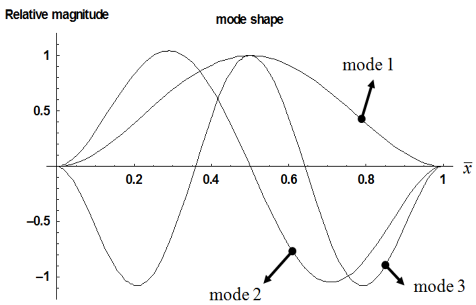

Fig. 2. Vibration modes of a free vibrating micro fixed-fixed beam

Tabulated in Table 2 are simulated dimensionless natural frequencies for respective modes and orders of differential transform. Illustrated in Fig. 3 is a plot of dimensionless natural frequencies against order k, from which a fast convergence is seen.

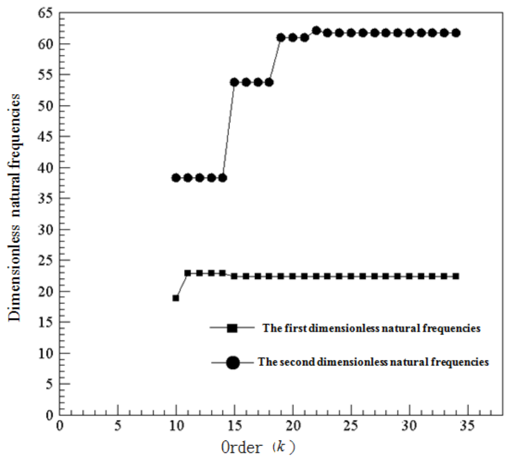

Fig. 3. A plot of dimensionless natural frequencies against order k for the micro fixed-fixed beam

Moreover, as compared in Table 1, a rather low relative error below 0.006 % is seen between the numerical and analytical solutions of dimensionless natural frequencies, that is, a good agreement in dealing with the eigenvalue problem of a micro system.

Table 1. Agreement between numerical and analytical solutions for dimensionless natural frequencies

	Numerical results (D. T. M.) (1)	Analytical results [16] (2)	Error (Δe) (%)
The first dimensionless natural frequencies	22.3733	22.3729	0.001788
The second dimensionless natural frequencies	61.6728	61.6696	0.005189

$$\Delta e = \frac{|(2)-(1)|}{(2)} \times 100\%$$

Table 2. Listing of the dimensionless natural frequencies against the order *k*

Order (*k*)	The first dimensionless natural frequencies	The second dimensionless natural frequencies
10.0	18.8206	x
11.0	22.8271	38.3542
12.0	22.8271	38.3542
13.0	22.8271	38.3542
14.0	22.8271	38.3542
15.0	22.3776	53.7016
16.0	22.3776	53.7016
17.0	22.3776	53.7016
18.0	22.3718	x
19.0	22.3733	60.9658
20.0	22.3733	60.9658
21.0	22.3733	60.9658
22.0	22.3733	62.1521
23.0	22.3733	61.6611
24.0	22.3733	61.6611
25.0	22.3733	61.6611
26.0	22.3733	61.6779
27.0	22.3733	61.6727
28.0	22.3733	61.6727
29.0	22.3733	61.6727
30.0	22.3733	61.6729
31.0	22.3733	61.6728
32.0	22.3733	61.6728
33.0	22.3733	61.6728
34.0	22.3733	61.6728

Pull-in Voltage Prediction of the Micro Cantilever Beam

The micro cantilever beam is shown schematically in Fig. 4 as being constructed from a single beam to a fixed frame. Following assumption have been supposed to simplify the analysis:

a. residual stress is ignored;
b. fringing field effect is ignored;
c. small deflection is assumed.

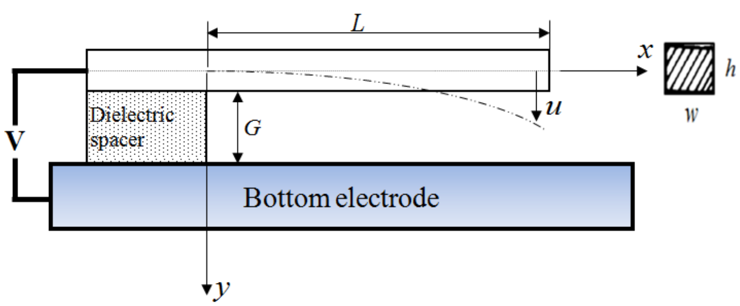

Fig. 4. A schematic diagram of the micro cantilever beam

The nonlinear governing equation of the micro cantilever beam is as follows [14]:

$$\tilde{E}I\frac{\partial^4 u}{\partial x^4}+\rho A\frac{\partial^2 u}{\partial t^2}=\frac{\varepsilon_0 w V^2}{2(G-u)^2} \tag{23}$$

The boundary condition is, for example:

$$\begin{aligned} &u(x,t)=\frac{\partial u(x,t)}{\partial x}=0 \quad \text{at} \quad x=0 \\ &\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{\partial^3 u(x,t)}{\partial x^3}=0 \quad \text{at} \quad x=L \end{aligned} \tag{24}$$

and the initial condition is, for example:

$$u(x,0)=\frac{\partial u(x,0)}{\partial t}=0 \tag{25}$$

Based on the assumption of the small deflection and adopting the Taylor series expansion, the electrostatic force term can be simplified as fifth-order model by omitting the terms of higher order. For analysis convenience, the nonlinear governing equation may be normalized. In order to analyze the pull-in behavior of the complex micro cantilever structure, the method of hybrid scheme [14], comprising the differential transformation method and the finite difference method, is used to obtain the numerical results of pull-in voltage for the micro cantilever beam.

The geometric and material properties of the micro cantilever beam are: Young's modulus *E*, 169 GPa; density ρ, 2.33×10^3 Kg/m^3; length of the beam *L*, 500 μm; width of the beam *w*, 1.6 μm; thickness of the beam *h*, 4.6 μm; initial gap between electrodes, 32.2 μm. The air permittivity of free space ε_0 is 8.85 PF/m and Poisson's ratio is 0.06. Table 3 shows the numerical results of pull-in voltage by hybrid method, which greatly resembled Adomian Decomposition Method [17] in the narrow micro cantilever beam.

Table 3. Comparison of simulated pull-in voltage with numerical results

Numerical method	Pull-in voltage (V)
Engery model [18]	112
Adomian decomposition method [17]	104
Hybrid method (H. M.)	108

Conclusions

In the present study, the natural frequencies of a micro fixed-fixed beam has been derived as the solutions to a boundary value problem with prescribed boundary conditions through the differential transformation method. As tabulated in Table 1, the numerical results are found in good agreement with analytical solutions, verifying the accuracy of this differential transform approach when applied to the natural frequency analysis problem of a micro fixed-fixed beam. In addition, a hybrid method that combines differential transformation and finite difference approximation has been demonstrated as a useful and powerful numerical analysis methodology. This method has been successfully applied to analyze the nonlinear partial differential equation of the micro cantilever beam. Briefly, the numerical results found for the pull-in voltage are close to the other literature results. Hence, the hybrid method applied in this paper also has great potential for analyzing other types of complex micro-structure devices in the future, such as the fringe effect and the residual stress effect.

Acknowledgements

The authors gratefully acknowledge the financial support provided to this study by the National Science Council of Taiwan, R. O. C. under Grant Number NSC 100-2221-E-018 -035.

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