908. Dynamic responses of an elastic beam moving over a simple beam using modal superposition method

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Abstract. The dynamic responses of an elastic beam moving over a simple beam are investigated. The elastic beam is modeled as an Euler beam with both ends free, and connected to the simple beam by two spring units. With modal superposition method, the dynamic responses of these two beams are studied. The vibrations of the simple beam are almost the same due to the moving elastic beam and rigid beam, even the latter beam ignores the flexural vibration of the elastic beam. However, the acceleration of the moving elastic beam is much larger than that of the moving rigid beam, which can be attributed to the flexible vibration of the elastic beam. With various flexural stiffness of the elastic beam, the max accelerations of the elastic beam at midpoint are computed along with different moving velocities. It is observed that, with an increase of flexural stiffness of the elastic beam, the max acceleration of the elastic beam decreases evidently.

Keywords: the elastic beam, the simple beam, flexible vibration, flexural stiffness.

1. Introduction

Traffic induced structural vibration has been an interesting topic in the field of civil engineering, such as railway and roadway bridge vibrations. Numerous researches on the dynamic behaviors of bridge under moving vehicle have been conducted [1-11]. In these studies, the moving vehicle is regarded as moving load [4, 5], moving mass [6, 7], moving oscillator [8, 9], and moving suspend beam considering pitching effect [10, 11].

As the demand of lightweight design for car structures, the structural stiffness of car frame will be decreased [12, 13]. To consider the flexible vibration of the moving structure, Zhang et al. modeled an elastic beam moving over another Euler beam with spring connected at two discrete points and some approximate analytical results were put forward [14]. Cojocaru et al. assumed a moving elastic beam connected to another beam with a series of rigid interfaces and the dynamic responses of this system were solved by means of symbolic computation [15, 16]. In the above mentioned work, some important conclusions have been brought out; while relatively little research attention so far seems to conduct the effect of flexural stiffness of the moving elastic beam on the dynamic responses of the system.

In the present paper, the dynamic responses of the elastic beam moving over a simple beam are investigated with modal superposition method. The elastic beam is regarded as an Euler beam with both ends free, and it is connected to the simple beam by two spring units. Firstly, the dynamic responses of these two beams are compared under the moving elastic and rigid beam. Secondly, with different flexural stiffness of the elastic beam, the max accelerations of the elastic beam at midpoint are computed along with different moving velocities. It is observed that the increase of flexural stiffness of the elastic beam can lead to a notable decrease of its max acceleration. From practical view, these results are useful for lightweight design of car structures to relief much more flexible vibration and improve riding comfort.

2. Formulations

In this paper, an elastic beam travels over a simple beam at a constant speed v, as shown in Fig. 1. In this model, the elastic beam is modeled as an Euler beam with both ends free, and connected to the simple beam by spring units at two discrete points. For the initial conditions, the rear spring unit of the elastic beam is located at the left-hand end of the simple beam.

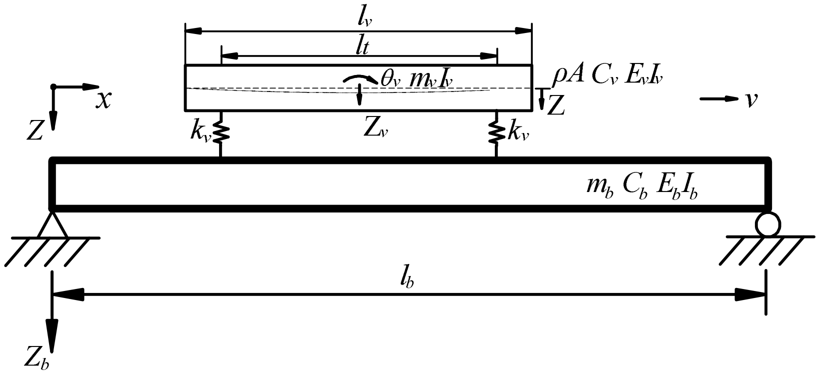

Fig. 1. An elastic beam moves over a simple beam

2.1 The elastic beam

As shown in Fig. 1, the vibration of the moving elastic beam is measured from its static equilibrium position, and its vibration can be divided into two parts: rigid vibration and flexible vibration [14]. The following symbols are used in Fig. 1: m_v = the total mass, I_v = the pitch of inertia, $E_v I_v$ = the flexural stiffness, C_v = the damping per unit length, ρA = the constant mass per unit length, l_v = the total length, l_t = the distance between the front and rear spring units, k_v = the stiffness of each spring unit. When the elastic beam runs on the simple beam, the equation of motion for the elastic beam can be written as:

$$E_v I_v \frac{\partial^4 Z(x_v,t)}{\partial x^4} + \rho A \frac{\partial^2 Z(x_v,t)}{\partial t^2} + C_v \frac{\partial Z(x_v,t)}{\partial t} = -\sum_{j=1}^{2} P_j \delta(x_v - l_j) \quad (1)$$

where $Z(x_v, t)$ is the vertical displacement of the elastic beam at time t ($0 \le x_v \le l_v$); $j = 1, 2$ represent the front and rear spring units, $\delta(\cdot)$ is the Dirac delta function, P_j are the interaction forces from the spring units:

$$P_j = k_v \left[Z(l_j,t) - Z_b(x_j,t) \right] \quad (2)$$

where $Z_b(x, t)$ is the vertical displacment of the simple beam, $l_j = l_v/2+(-1)^{j+1} l_t/2$, x_j is the distance of the front/rear spring units to the left-hand of the simple beam.

The vibration of the elastic beam can be regarded as the combination of rigid motion and flexible motion. When the rigid modes are included in the flexible modes, the first mode is chosen as the bounce of rigid mode and its shape function is taken as $Y_1(x_v) = 1$. The second mode is the pitch and its shape function is $Y_2(x_v) = x_v - l_v/2$. When NM_v modes are considered, the vertical displacement of the elastic beam can be written as [17]:

$$Z(x_v,t) = Z_v(t) + \left(x_v - \frac{l_v}{2}\right)\theta_v(t) + \sum_{m=3}^{NM_v} p_m(t) Y_m(x_v) \tag{3}$$

where $Z_v(t)$ and $\theta_v(t)$ are the modal amplitude of the bounce and pitch motion respectively, $Y_m(x_v)$ and $p_m(t)$ are the modal amplitude and mode shape function of the flexible vibration for the elastic beam.

Consider a beam with both ends free, when $m > 2$, the mode shape functions of the elastic beam can be given as [18]:

$$Y_m(x_v) = \cosh(\beta_m x_v) + \cos(\beta_m x_v) - \frac{\cosh\lambda_m - \cos\lambda_m}{\sinh\lambda_m - \sin\lambda_m}\left[\sinh(\beta_m x_v) + \sin(\beta_m x_v)\right] \tag{4}$$

where λ_m and β_m satisfy:

$$1 - \cosh\lambda_m \cos\lambda_m = 0,\ \ \beta_m = \lambda_m / l_v \tag{5}$$

Substituting Eq. (3) into Eq. (1), and multiplying by $Y_i(x_v)$ and integrating the resultant equation with respect to x_v between 0 and l_v, and considering the orthogonality conditions of the natural vibration modes, the equations of motion in terms of the NM_v modal displacements can be given as:

$$m_v \ddot{Z}_v(t) + \sum_{j=1}^{2} P_j = 0 \tag{6}$$

$$I_v \ddot{\theta}_v(t) + \frac{l_t}{2}\sum_{j=1}^{2} (-1)^{j+1} P_j = 0 \tag{7}$$

$$m_v \ddot{p}_i(t) + \frac{m_v C_v}{\rho A}\dot{p}_i(t) + \frac{m_v E_v I_v \beta_i^4}{\rho A} p_i(t) = -\sum_{j=1}^{2} Y_i(l_j) P_j \quad i = 3 \cdots NM_v \tag{8}$$

Let $\frac{E_v I_v \beta_i^4}{\rho A} = \omega_{vi}^2$, $\frac{C_v}{\rho A} = 2\xi_{vi}\omega_{vi}$, the Eq. (8) can be given as:

$$m_v \ddot{p}_i(t) + 2\xi_{vi}\omega_{vi} m_v \dot{p}_i(t) + m_v \omega_{vi}^2 p_i(t) = -\sum_{j=1}^{2} Y_i(l_j) P_j \quad i = 3 \cdots NM_v \tag{9}$$

Therefore Eqs. (6), (7) and (9) are expressed for the rigid motion and flexible motion of the elastic beam.

2.2 The simple beam

As shown in Fig. 1, the equation of motion for the simple beam due to moving elastic beam can be written as:

$$E_b I_b \frac{\partial^4 Z_b(x,t)}{\partial x^4} + m_b \frac{\partial^2 Z_b(x,t)}{\partial t^2} + C_b \frac{\partial Z_b(x,t)}{\partial t} = F_b(x,t) \tag{10}$$

where E_bI_b is the flexural stiffness, m_b is the constant mass per unit length, C_b is the damping per unit length, $F_b(x, t)$ is the external force acting on the simple beam from the elastic beam:

$$F_b\left(x,t\right)=\sum_{j=1}^{2}\left(\frac{1}{2}m_v g+P_j\right)\cdot\delta\left(x-x_j\right) \tag{11}$$

Based on the modal superposition method, the solution of Eq. (10) can be expressed as:

$$Z_b(x,t)=\sum_{n=1}^{NM_b}q_n\left(t\right)\phi_n\left(x\right) \tag{12}$$

where $q_n(t)$ is the nth modal amplitude and $\phi_n(x)$ is the nth mode shape function of the beam.

For simple beam, the natural frequency and mode shape functions can be expressed as:

$$\omega_n=\left(\frac{n\pi}{l_b}\right)^2\sqrt{\frac{E_bI_b}{m_b}} \tag{13}$$

$$\phi_n\left(x\right)=\sin\left(\frac{n\pi x}{l_b}\right) \tag{14}$$

Substituting Eqs. (11) and (12) into Eq. (10), and multiplying by $\phi_k(x)$ and integrating the resultant equation with respect to x between 0 and l_b, and considering the orthogonality conditions of the natural vibration modes, the equation of motion of the kth generalized system in terms of the generalized displacement $q_k(t)$ is given as:

$$M_k\ddot{q}_k\left(t\right)+2\xi_k\omega_k M_k\dot{q}_k\left(t\right)+K_kq_k\left(t\right)=F_k\left(t\right) \tag{15}$$

where ω_k, ξ_k and M_k are the modal frequency, damping ratio, and modal mass of the kth mode, respectively, and $K_k\left(=M_k\omega_k^2\right)$ means the geneneralized stiffness of the kth mode. The generalized force $F_k(t)$ is expressed as:

$$F_k\left(t\right)=\sum_{j=1}^{2}\left(\frac{1}{2}m_v g+P_j\right)\cdot\phi_k\left(x_j\right) \tag{16}$$

3. Solution

Subsequently, substituting Eqs. (2), (3), (12) into Eqs. (6), (7), (9) and moving the terms with $\left(Z_v,\theta_v,p_m,q_n\right)$ to the left side of the differential equation, the equations of motion for the elastic beam can be written as:

$$m_v\ddot{Z}_v\left(t\right)+2k_vZ_v\left(t\right)+k_v\sum_{m=3}^{NM_v}\sum_{j=1}^{2}Y_m\left(l_j\right)p_m\left(t\right)-k_v\sum_{n=1}^{NM_b}\sum_{j=1}^{2}\phi_n\left(x_j\right)q_n\left(t\right)=0 \tag{17}$$

$$I_v\ddot{\theta}_v(t)+\frac{k_v l_t^2}{2}\theta_v(t)+\frac{k_v l_t}{2}\sum_{m=3}^{NM_v}\sum_{j=1}^{2}(-1)^{j+1}Y_m(l_j)p_m(t)-\frac{k_v l_t}{2}\sum_{n=1}^{NM_b}\sum_{j=1}^{2}(-1)^{j+1}\phi_n(x_j)q_n(t)=0 \quad (18)$$

$$m_v\ddot{p}_i(t)+2\xi_{vi}\omega_{vi}m_v\dot{p}_i(t)+\omega_{vi}^2 m_v p_i(t)+k_v\sum_{j=1}^{2}Y_i(l_j)Z_v(t)+\frac{k_v l_t}{2}\sum_{j=1}^{2}(-1)^{j+1}Y_i(l_j)\theta_v(t)$$
$$+k_v\sum_{j=1}^{2}\sum_{m=3}^{NM_v}Y_i(l_j)Y_m(l_j)p_m(t)-k_v\sum_{j=1}^{2}\sum_{n=1}^{NM_b}Y_i(l_j)\phi_n(x_j)q_n(t)=0 \quad (19)$$

Substituting Eqs. (2), (3), (12), (16) into Eq. (15) and moving the terms with (Z_v,θ_v,p_m,q_n) to the left side of the differential equation, the equation of motion for the simple beam can be written as:

$$M_k\ddot{q}_k(t)+2\xi_k\omega_k M_k\dot{q}_k(t)+K_k q_k(t)+k_v\sum_{j=1}^{2}\sum_{n=1}^{NM_b}q_n(t)\Phi_{nk}-k_v\sum_{j=1}^{2}Z_v(t)\Phi_k$$
$$-\frac{k_v l_t}{2}\sum_{j=1}^{2}(-1)^{j+1}\theta_v(t)\Phi_k-k_v\sum_{j=1}^{2}\sum_{m=3}^{NM_v}p_m(t)Y_m(l_j)\Phi_k=\frac{m_v g}{2}\sum_{j=1}^{2}\Phi_k \quad (20)$$

where $\Phi_{nk}=\phi_n(x_j)\phi_k(x_j)$ and $\Phi_k=\phi_k(x_j)$.

As shown in Eqs. (17), (18), (19) and (20), the elastic beam and the simple beam are coupled and interacting with each other. By combining Eqs. (17), (18), (19) and (20) together, the equations of motion in modal space are given in a matrix form as:

$$[M]\{\ddot{U}\}+[C]\{\dot{U}\}+[K]\{U\}=\{F\} \quad (21)$$

where [M], [C], [K] denote the mass, damping and stiffness matrices; $(\{U\},\{\dot{U}\},\{\ddot{U}\})$ are the vectors of displacement, velocity, and acceleration, respectively; and {F} represents the vector of exciting forces applied to the dynamic system.

To compute both the dynamic responses of the simple beam and the elastic beam, the equations of motion as given in Eq. (21) will be solved using a step-by-step integration method, i.e. Newmark-β method [19]. The integration scheme of Newmark-β method consists of the following equations:

$$\{\ddot{U}\}_{t+\Delta t}=a_0\left(\{U\}_{t+\Delta t}-\{U\}_t\right)-a_2\{\dot{U}\}_t-a_3\{\ddot{U}\}_t \quad (22)$$

$$\{\dot{U}\}_{t+\Delta t}=\{\dot{U}\}_t+a_6\{\ddot{U}\}_t+a_7\{\ddot{U}\}_{t+\Delta t} \quad (23)$$

where the coefficients are:

$$\alpha_0=\frac{1}{\beta\Delta t^2},\ \alpha_1=\frac{\gamma}{\beta\Delta t},\ \alpha_2=\frac{1}{\beta\Delta t},\ \alpha_3=\frac{1}{2\beta}-1,$$
$$\alpha_4=\frac{\gamma}{\beta}-1,\ \alpha_5=\frac{\Delta t}{2}(\frac{\gamma}{\beta}-2),\ \alpha_6=\Delta t(1-\gamma),\ \alpha_7=\gamma\Delta t. \quad (24)$$

In this study, β = 1/4 and γ = 1/2 are selected, which implies a constant acceleration with unconditional numerical stability.

4. Numerical investigation

Fig. 1 shows an elastic beam crossing a simple beam at a constant speed v, and the properties of these two beams are listed in Table 1 and Table 2, where ω_v and ω_b denote the first natural frequencies of the elastic beam and the simple beam, respectively. In the illustrative example, a time step of 0.0001 s and ending time of $t_{end} = (l_b - l_t)/v$ are employed to compute the dynamic responses of the elastic beam and the simple beam.

Table 1. Properties of the elastic beam

m_v (kg/m)	l_v (m)	l_t (m)	E_vI_v (kN m^2)	I_v (kg m^2)	k_v (N m^{-1})	ξ_v	ω_v (Hz)
18000	3.6	3.0	2.6×10^4	7.2×10^4	3.6×10^6	1.5 %	19.81

Table 2. Properties of the simple beam

m_b (kg/m)	l_b (m)	E_bI_b (kN m^2)	ξ_b	ω_b (Hz)
28125	15	1.8×10^7	2.5 %	5.59

To compute the dynamic response of the simple beam under the moving elastic beam, a sufficient number of modes in Eqs. (3) and (12) are required for accuracy of the response computed from Eq. (21). For the simple beam, 20 modes are sufficient to determine the dynamic response of the simple beam [20], which are also used in this paper. For the elastic beam, in order to verify that a sufficient number of modes has been used in the analysis, we first compute the acceleration response of the elastic beam at midpoint using either 8, 14, or 20 modes (NM_v = 10, 16, or 22) with the moving speed at 15 m/s.

From the convergent verification of computed results in Fig. 2, the first 20 modes (NM_v = 22) are sufficient to compute the acceleration response of the elastic beam moving over the simple beam. For this reason, the same number of modes will be considered in the following examples.

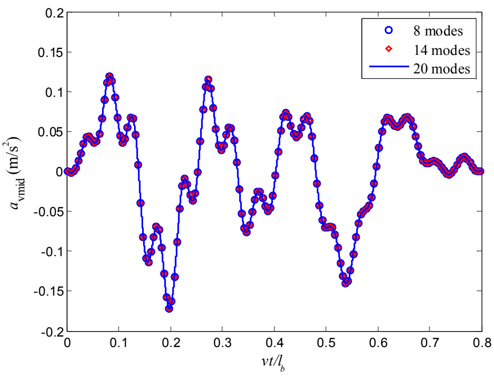

Fig. 2. Test of convergence

4.1 Comparison of the moving elastic and rigid beam

Firstly, the moving beam is regarded as an elastic beam and a rigid beam, respectively. When the moving beam runs over the simple beam at 15 m/s, the displacement responses of the simple beam at midpoint are computed, as shown in Fig. 3. It can be noticed that the displacement responses of the simple beam at midpoint under the moving elastic and rigid beam are almost the same. It also means that the effect of flexibility of the moving beam on the vibration of the simple beam can be negligible, and this is because the forces acting on the simple beam have little variation from the gravity of the moving beam [21].

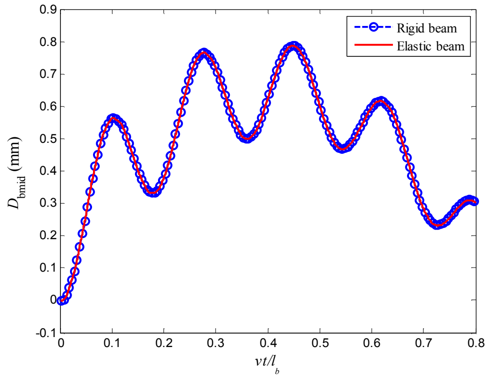

Fig. 3. Displacement responses of the simple beam at midpoint

When the moving beam runs over the simple beam at 15 m/s, the acceleration responses of the elastic and rigid beam at midpoint are computed, as shown in Fig. 4. It can be seen that the accleeration response of the elastic beam at midpoint is a little larger than that of the rigid beam, which can be attritubted to the flexible vibration of the elastic beam excited by bridge vibration. This phenomean is confirmed by the acceleration spetruem analysis, as shown in Fig. 5. The first dominant frequencies of the moving elastic and rigid beam are 5 Hz, which is close to the natural frequency of the simple beam (See Table 2). Besides the first dominant frequency at 5 Hz, there is another dominant frequency for the moving elastic beam at 20 Hz, which is consitant with the natural frequency of the elastic beam (See Table 1).

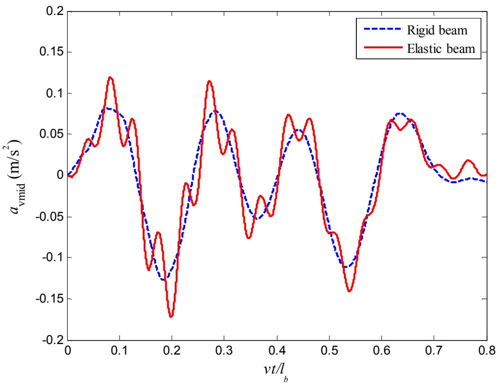

Fig. 4. Acceleration responses of the elastic beam at midpoint

4.2 Effect of different flexural stiffness

Acctually, the flexible vibration of the elastic beam is excited by bridge vibration as indicated in Section 4.1, and the effect of different flexural stiffness on the vibration of the elastic beam will be performed by considering various flexural stiffness, i.e., $E_vI_v/10$, E_vI_v, and $10E_vI_v$. Then, the max accerelations of the elastic beam at midpoint are plotted against the moving velocities at 5, 10, 15, 20, and 25 m/s in Fig. 6. And the results are also compared with the moving rigid beam.

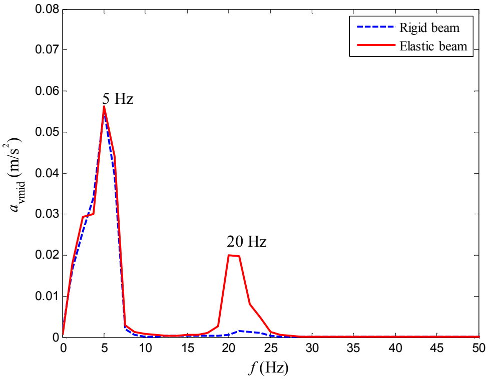

Fig. 5. Accelerations spectrum of the elastic beam at midpoint

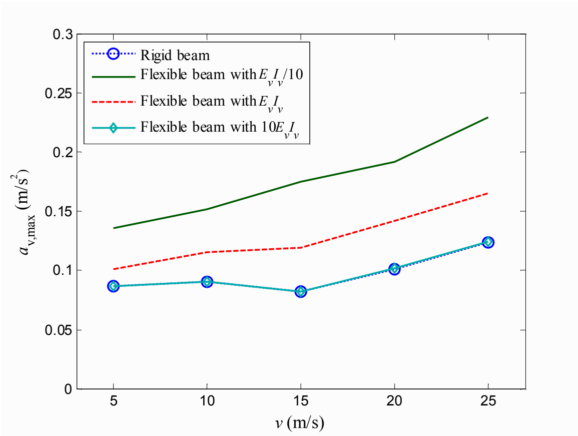

Fig. 6. Max accelerations of the elastic beam at midpoint vs moving velocities

It can be seen from the Fig. 6 that, with the increase of flexural stiffness of the elastic beam, the max acceleration of the elastic beam at midpoint decreases observably. In this numerical example, when the flexural stiffness of the elastic beam is eaqual to $10E_vI_v$, resulting in the natural frequency of 62.65 Hz, the max acceleration of the elastic beam is almost the same with the rigid beam. Therefore, to supress the flexible vibration of the elastic beam, its flexural stiffness should be improved at a certain value, which is important for the lightweight design of car structures.

5. Conclusions

In this paper, the dynamic responses of an elastic beam moving over a simple beam are investigated using modal superposition method. In this model, the elastic beam is modeled as an Euler beam with both ends free, and connected to the simple beam by two spring units. On the base of this study, the following conclusions can be drawn:

(1) The displacements of the simple beam at midpoint are almost the same under the moving elastic and rigid beam, whereas the acceleration of the moving elastic beam at midpoint is a little larger than that of the moving rigid beam. This can be attributed to the flexible vibration of the elastic beam excited by simple beam vibration.

(2) With various flexural stiffness of the elastic beam, the max accelerations of the elastic beam at midpoint are computed along with different moving velocities. It is observed that an increase of the flexural stiffness of the elastic beam can lead to a notable decrease of its max acceleration. From

practical view, the results are useful for lightweight design of car structures to relief much more flexible vibration and improve riding comfort.

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