

# 1187. Self-synchronization theory of a nonlinear vibration system driven by two exciters. Part 1: Theoretical analysis

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**Abstract.** A single mass vibration system driven by two counter-rotating motors is studied in this paper. The nonlinear factor of vibration spring in the vertical direction is considered. Based on the Lagrange Equations, the mechanic-electric coupling dynamic equation of the vibration system is established. The condition of the self-synchronization implementation of the system is obtained by using the Hamilton theory. Applying once approximate method of nonlinear system stability, the asymptotic stability condition of the vibration system at equilibrium point is deduced. They provide theoretical basis for the simulation of self-synchronization of vibration system.

**Keywords:** nonlinear, vibration system, self-synchronization, synchronization stability.

## 1. Introduction

The earliest detailed accounts on synchronized motion was made by Huygens [1], who observed that two clock pendulums suspended from stiff wooden beams could run in a steady state and move in opposition to each other at the same angular velocity. Since 1894, the synchronous phenomena was also found in nonlinear circuits by scientists, such as Rayleigh [2] found that two organ tubes could produce a synchronized sound when the outlets were close to each other and Pol [3] observed the synchronization of certain electrical-mechanical system. They called this phenomenon as “frequency capture”, which was a unique phenomenon of nonlinear system. In the 1960s, Dr. Blekhnman I. I. [4-6] in Soviet Union proposed the self-synchronization theory of vibrating machinery with double exciters and established the conditions of existence and stability of self-synchronization of the exciters in vibrating system based on the analytical method of direct motion separation. The method was applied later by some researchers Ragulskis K. M. [7-8], Khodzhaev K. Sh. [9], Sperling L. [1-11] and Nagaev R. F. [12] from Lithuania, Russia, Germany and other countries to a number of synchronization problems. It has been proven to be useful and descriptive leading to better understanding and theoretical explanation of the mechanism of self-synchronization. Japanese researchers Inoue J. and Araki Y. [13] et al. studied the multiple frequency self-synchronization of vibration machine driven by two motors. Chinese scholar Prof. Wen [14-16], selected the phase difference between two exciters as the variable to simplify the analytical method for establishing the conditions of existence and stability of self-synchronization of two identical exciters in a vibrating system. Later Balthazar J. M. [17] from Brail has also given some short comments on self-synchronization of two non-ideal sources. Yamapi R. [18-19] in Cameroon introduced dynamic characteristics of two motors to the self-synchronization theory. The self-synchronization of a vibrating system from the effect of electric-mechanic motors coupling was dependent on the dynamic parameters of two induction motors. In recent years, many scholars [20-22] considered nonlinear factor of vibration system and the vibration problems of nonlinear system were solved well.

In this paper, a single-mass nonlinear vibrating system driven by two counter-rotating motors is studied. The nonlinear factor of vibration spring in the vertical direction is considered. A mechanic-electric coupling dynamic model of the vibration system is established. The condition

of self-synchronous operation implementation of the vibration system is obtained according to Hamilton principle. Based on once approximate method of stability of nonlinear system, the asymptotic stability condition of the vibration system at the equilibrium point is deduced. They provide theoretical basis for the simulation of self-synchronization of vibration system.

## 2. The mechanic-electric coupling dynamic model of the vibrating system

The mechanical model of a single-mass nonlinear vibrating system driven by two counter-rotating motors is presented in Fig. 1, which consists of a vibrating body and two eccentric rotors. The vibrating body is connected with fixed support by four springs in  $x$ -axis and  $y$ -axis directions, respectively. The hard characteristic of the springs in  $y$ -axis direction is considered. Fig. 2 shows the hard characteristic curve. The expression of elastic force is described as Eq. (1):

$$F = kx + k'x^3. \tag{1}$$

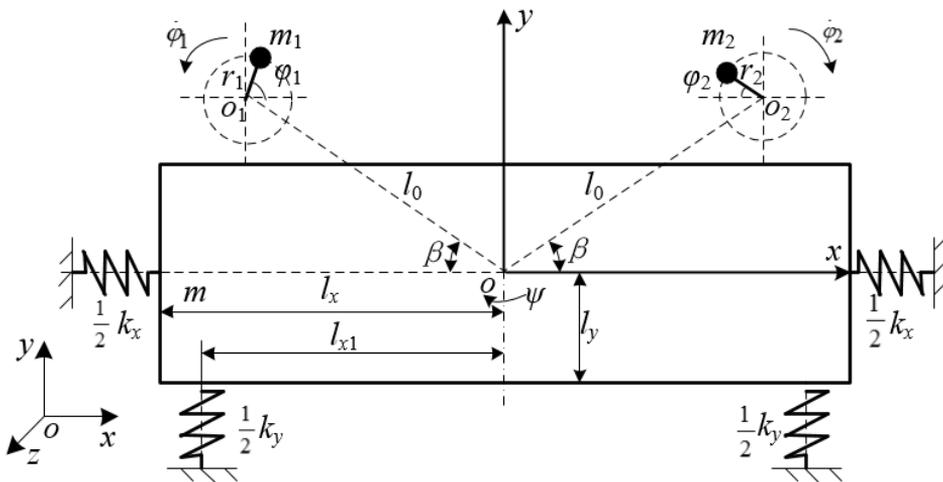


Fig. 1. The mechanical model of a single-mass nonlinear vibration system

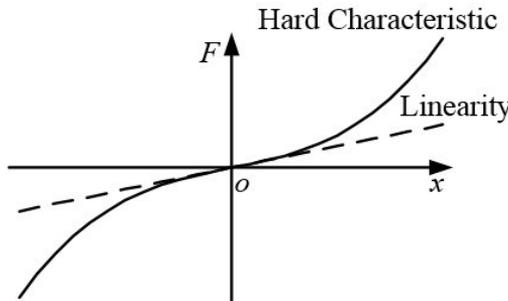


Fig. 2. Hard characteristic curve of nonlinear spring

The two eccentric rotors driven by two induction motors separately are installed symmetrically about the  $yo$  $z$  plane which goes through the centroid of the vibrating body.  $o_1$  and  $o_2$  are separately the rotation centers of the two counter-rotating eccentric rotors.

As can be seen from Fig. 1, the vibrating body exhibits three degrees of freedom, i.e., two translation motions in the  $x$ -axis and  $y$ -axis directions and one swing in  $\psi$  direction. In addition, the eccentric rotor 1 and 2 rotate about their own spindle and the rotor angle are separately denoted by  $\phi_1$  and  $\phi_2$ . Therefore, the vibrating system needs five independent coordinates to describe its special position, i.e., this vibrating system has five degrees of freedom.

As illustrated in Fig. 3, three reference frames can be assigned as follows: the fixed coordinate system  $oxy$  which is fixed to the inertial space; the translation coordinate system  $o'x'y'$  and the rotating coordinate system  $o'x''y''$  which are both fixed to the vibrating body. In the rotating coordinate system  $o'x''y''$ , the centroid coordinates of the eccentric rotor 1 and 2 can be expressed as:

$$\begin{cases} x_1'' \\ y_1'' \end{cases} = \begin{cases} x + r_1 \cos(\varphi_1 + \psi) - l_0 \cos(\beta - \psi) \\ y + r_1 \sin(\varphi_1 + \psi) + l_0 \sin(\beta - \psi) \end{cases}, \quad (2)$$

$$\begin{cases} x_2'' \\ y_2'' \end{cases} = \begin{cases} x - r_2 \cos(\varphi_2 - \psi) + l_0 \cos(\beta + \psi) \\ y + r_2 \sin(\varphi_2 - \psi) + l_0 \sin(\beta + \psi) \end{cases},$$

where  $r_1$  and  $r_2$  are the rotation radius of the eccentric rotor 1 and 2, respectively;  $l_0$  is the distance between the center of rotation of each eccentric rotor and the centroid of the vibration body;  $\beta$  is the included angle between the linkage between the center of rotation of each eccentric rotor and the centroid of the vibration body and horizon  $x$ -axis.  $\varphi_1$  and  $\varphi_2$  are separately the rotor angles of the eccentric rotor 1 and 2;  $\psi$  is the rotation angle of the vibration body around the  $z$ -axis rotation.

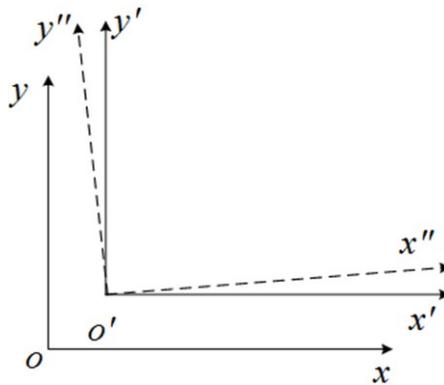


Fig. 3. Coordinate systems: fixed-coordinate system  $oxy$ ; translation coordinate system  $o'x'y'$ ; rotating coordinate system  $o'x''y''$

The kinetic energy  $T$  of the vibrating system can be expressed as follows:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\psi}^2 + \frac{1}{2}m_1(\dot{x}_1''^2 + \dot{y}_1''^2) + \frac{1}{2}m_2(\dot{x}_2''^2 + \dot{y}_2''^2) + \frac{1}{2}J_{01}\dot{\varphi}_1^2 + \frac{1}{2}J_{02}\dot{\varphi}_2^2, \quad (3)$$

where  $m$  is the mass of vibration body;  $m_1$  and  $m_2$  are the mass of exciter 1 and 2;  $J$  is the moment of inertia of vibration body;  $J_{01}$  and  $J_{02}$  are the moments of inertia of motor 1 and 2.

The vibrating body is connected with fixed base by four springs in  $x$ -axis and  $y$ -axis directions, respectively. When the vibration system is working, the deformations of springs A, B, C and D are expressed as:

$$\begin{aligned} \Delta x_A &= |x + l_x(1 - \cos\psi)|, & \Delta y_A &= |y - l_x \sin\psi|, \\ \Delta x_B &= |x - l_x(1 - \cos\psi)|, & \Delta y_B &= |y + l_x \sin\psi|, \\ \Delta x_C &= |x + l_{x1}(1 - \cos\psi) + l_y \sin\psi|, & \Delta y_C &= |y - l_{x1} \sin\psi + l_y(1 - \cos\psi)|, \\ \Delta x_D &= |x - l_{x1}(1 - \cos\psi) + l_y \sin\psi|, & \Delta y_D &= |y + l_{x1} \sin\psi + l_y(1 - \cos\psi)|. \end{aligned} \quad (4)$$

Then the potential energy  $V$  of the system can be expressed as:

$$V = \frac{1}{2}k_A(\Delta x_A^2 + \Delta y_A^2) + \frac{1}{2}k_B(\Delta x_B^2 + \Delta y_B^2) + \frac{1}{2}k_C(\Delta x_C^2 + \Delta y_C^2) + \frac{1}{2}k_D(\Delta x_D^2 + \Delta y_D^2) + \frac{1}{2}k'_y f^2(y, \dot{y}), \quad (5)$$

where  $k_A$  and  $k_B$  are the stiffness coefficient of spring A and B,  $k_A = k_B = k_x/2$ ;  $k_C$  and  $k_D$  are the stiffness coefficient of spring C and D,  $k_C = k_D = k_y/2$ ;  $k'_y$  is the stiffness coefficient of nonlinear spring, which can be obtained by experimental curve of spring;  $f(y, \dot{y})$  is the nonlinear function in  $y$ -axis direction,  $f(y, \dot{y}) = y^3$ .

Then the viscous dissipation function  $D$  of the system can be described as the following:

$$D = \frac{1}{2}f_A(\Delta \dot{x}_A^2 + \Delta \dot{y}_A^2) + \frac{1}{2}f_B(\Delta \dot{x}_B^2 + \Delta \dot{y}_B^2) + \frac{1}{2}f_C(\Delta \dot{x}_C^2 + \Delta \dot{y}_C^2) + \frac{1}{2}f_D(\Delta \dot{x}_D^2 + \Delta \dot{y}_D^2), \quad (6)$$

where  $f_A, f_B, f_C$  and  $f_D$  are the damping coefficient of spring A, B, C and D, respectively.

The Lagrange's equation is expressed as:

$$\frac{d}{dt} \frac{\partial(T - V)}{\partial \dot{q}_i} - \frac{\partial(T - V)}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i, \quad (7)$$

where  $q_i$  is the generalized coordinate of the system;  $Q_i$  is the generalized force of the system.  $q = [x, y, z, \psi, \varphi_1, \varphi_2]^T$  is chosen as the generalized coordinates of the vibrating system and the generalized forces are:  $Q_x = Q_y = Q_\psi = 0$ ,  $Q_{\varphi_1} = T_{e1} - f_{d1}\dot{\varphi}_1$  and  $Q_{\varphi_2} = T_{e2} - f_{d2}\dot{\varphi}_2$ ;  $T_{e1}$  and  $T_{e2}$  are the electromagnetic torque of motor 1 and 2;  $f_{d1}$  and  $f_{d2}$  are the damping coefficient of motor 1 and 2.

Substitute Eqs. (3), (5) and (6) to Eq. (7) and the electromechanical-coupled dynamic model of the system can be reduced as follows:

$$\begin{aligned} M_a \ddot{x} + f_{xy} \dot{x} + k_{xy} x &= m_1 r_1 (\dot{\varphi}_1^2 \cos \varphi_1 + \ddot{\varphi}_1 \sin \varphi_1) - m_2 r_2 (\dot{\varphi}_2^2 \cos \varphi_2 + \ddot{\varphi}_2 \sin \varphi_2), \\ M_a \ddot{y} + f_{xy} \dot{y} + k_{xy} y + k'_y y^3 &= m_1 r_1 (\dot{\varphi}_1^2 \sin \varphi_1 - \ddot{\varphi}_1 \cos \varphi_1) + m_2 r_2 (\dot{\varphi}_2^2 \sin \varphi_2 - \ddot{\varphi}_2 \cos \varphi_2), \\ J_a \ddot{\psi} + f_\psi \dot{\psi} + k_\psi \psi &= -m_1 r_1 [l_0 \dot{\varphi}_1^2 \sin(\varphi_1 + \beta) + (r_1 - l_0 \cos(\varphi_1 + \beta)) \ddot{\varphi}_1] \\ &\quad + m_2 r_2 [l_0 \dot{\varphi}_2^2 \sin(\varphi_2 + \beta) + (r_2 - l_0 \cos(\varphi_2 + \beta)) \ddot{\varphi}_2], \\ (J_{o1} + m_1 r_1^2) \ddot{\varphi}_1 + f_{d1} \dot{\varphi}_1 &= T_{e1} \\ &\quad - m_1 r_1 [\dot{y} \cos(\varphi_1 + \psi) - \ddot{x} \sin(\varphi_1 + \psi) + \ddot{\psi} (r_1 - l_0 \cos(\varphi_1 + \beta)) - l_0 \dot{\psi}^2 \sin(\varphi_1 + \beta)], \\ (J_{o2} + m_2 r_2^2) \ddot{\varphi}_2 + f_{d2} \dot{\varphi}_2 &= T_{e2} \\ &\quad - m_2 r_2 [\dot{y} \cos(\varphi_2 - \psi) + \ddot{x} \sin(\varphi_2 - \psi) - \ddot{\psi} (r_2 - l_0 \cos(\varphi_2 + \beta)) - l_0 \dot{\psi}^2 \sin(\varphi_2 + \beta)], \end{aligned} \quad (8)$$

where  $M_a$  is the mass of vibration system,  $M_a = m + m_1 + m_2$ ;  $J_a$  is the moments of inertia of the vibration system,  $J_a = J + (m_1 + m_2)(l_0^2 + r^2)$ ;  $k_{xy}$  and  $k_\psi$  are the stiffness coefficients of the vibration system in  $x$ ,  $y$  and  $\psi$  directions,  $k_{xy} = k_x + k_y$  and  $k_\psi = k_y l_{x1}^2 + k_y l_y^2$ ;  $f_{xy}$  and  $f_\psi$  are the damping coefficients of the vibration system in  $x$ ,  $y$  and  $\psi$  directions,  $f_{xy} = f_x + f_y$  and  $f_\psi = f_x l_x^2 + f_y (l_{x1}^2 + l_y^2)$ ;  $(\ddot{\quad})$  denotes  $d^2(\quad)/dt^2$ .

According to the electromagnetic property of asynchronous motor [23], the relationship between electromagnetic torque  $T_e$  and speed  $n$  is deduced as Eq. (9):

$$T_e = 2T_{max} s_m n_N \frac{n_N - n}{s_m^2 n_N^2 + (n_N - n)^2}, \quad (9)$$

where  $T_{max}$  is the maximum torque;  $T_{max} = K_T T_N$ ;  $s_m$  is the critical slip,  $s_m = s_N (K_T + \sqrt{K_T^2 - 1})$ ;  $K_T$  is the overload coefficient;  $T_N$  is the rated torque;  $s_N$  is the rated

slip;  $n_N$  is the rated speed;  $n$  is the speed of motor at any time.

### 3. The condition of self-synchronization implementation

We assume that the average phase of two eccentric rotors is  $\varphi$ , and  $\varphi = \bar{\omega}t$ , where  $\bar{\omega}$  is the average angular velocity. The phase difference between the two eccentric rotors is  $2\alpha$ , i.e.:

$$2\varphi = \varphi_1 + \varphi_2, \quad 2\alpha = \varphi_1 - \varphi_2. \quad (10)$$

Then the phase of the eccentric rotor 1 and 2 are separately described as:

$$\varphi_1 = \varphi + \alpha = \bar{\omega}t + \alpha, \quad \varphi_2 = \varphi - \alpha = \bar{\omega}t - \alpha. \quad (11)$$

In the first three formulas of Eq. (8), the angle accelerations  $\ddot{\varphi}_1, \ddot{\varphi}_2$  of the eccentric rotor 1 and 2 are so small that they could be neglected. In actual project,  $m_1r_1 \approx m_2r_2$ . Therefore, the first three formulas of the electromechanical coupling dynamic equation Eq. (8) can be simplified as Eq. (12):

$$\begin{aligned} M_a \ddot{x} + f_{xy} \dot{x} + k_{xy} x &= -(m_1r_1 + m_2r_2) \dot{\varphi}^2 \sin\alpha \sin\varphi, \\ M_a \ddot{y} + f_{xy} \dot{y} + k_{xy} y + k'_y y^3 &= (m_1r_1 + m_2r_2) \dot{\varphi}^2 \cos\alpha \sin\varphi, \\ J_a \ddot{\psi} + f_\psi \dot{\psi} + k_\psi \psi &= -(m_1r_1 + m_2r_2) l_0 \dot{\varphi}^2 \sin\alpha \cos(\varphi + \beta). \end{aligned} \quad (12)$$

Introduce the following parameters:

$$\begin{aligned} \omega_x &= \sqrt{\frac{k_{xy}}{M_a}}, \quad \omega_y = \sqrt{\frac{k_{xy}}{M_a}}, \quad \omega_\psi = \sqrt{\frac{k_\psi}{J_a}}, \\ \xi_x &= \frac{f_{xy}}{2M_a\omega_x}, \quad \xi_y = \frac{f_{xy}}{2M_a\omega_y}, \quad \xi_\psi = \frac{f_\psi}{2J_a\omega_\psi}, \end{aligned} \quad (13)$$

where  $\omega_x, \omega_y$  and  $\omega_\psi$  are the natural frequency of the vibration system in  $x, y$  and  $\psi$  directions;  $\xi_x, \xi_y$  and  $\xi_\psi$  are the damping ratios of the vibration system in  $x, y$  and  $\psi$  directions.

For Eq. (12), the vibration response in each direction can be solved by gradual method of nonlinear system. When the system is working in the non-resonant case, the meaningful solution for project can be expressed as:

$$\begin{aligned} x &= a_x \sin(\bar{\omega}t - \vartheta_x), \\ y &= a_y \sin(\bar{\omega}t - \vartheta_y), \\ \psi &= a_\psi \cos(\bar{\omega}t + \beta - \vartheta_\psi), \end{aligned} \quad (14)$$

where the vibration amplitude, the lagging phase angle in each direction are separately:

$$\begin{aligned} a_x &= \frac{-(m_1r_1 + m_2r_2) \bar{\omega}^2 \sin\alpha}{M_a \sqrt{(\omega_x^2 - \bar{\omega}^2)^2 + (2\xi_x \omega_x \bar{\omega})^2}}, \quad \vartheta_x = \arctan \frac{2\xi_x \omega_x \bar{\omega}}{\omega_x^2 - \bar{\omega}^2}, \\ a_y &= \frac{(m_1r_1 + m_2r_2) \bar{\omega}^2 \cos\alpha}{M_a \sqrt{(\omega_y^2 - \bar{\omega}^2)^2 + (2\xi_y \bar{\omega})^2}}, \quad \vartheta_y = \arctan \frac{2\xi_y \bar{\omega}}{\omega_y^2 - \bar{\omega}^2}, \\ a_\psi &= \frac{-(m_1r_1 + m_2r_2) l_0 \bar{\omega}^2 \sin\alpha}{J_a \sqrt{(\omega_\psi^2 - \bar{\omega}^2)^2 + (2\xi_\psi \omega_\psi \bar{\omega})^2}}, \quad \vartheta_\psi = \arctan \frac{2\xi_\psi \omega_\psi \bar{\omega}}{\omega_\psi^2 - \bar{\omega}^2}, \end{aligned} \quad (15)$$

in which the equivalent natural frequency and the equivalent damping ratio in  $y$  direction are:

$$\omega_{ey} = \omega_y + \frac{3k'_y}{8M_a\omega_y}a_y^2, \quad \delta_{ey} = \frac{f_{xy}}{M_a}. \quad (16)$$

In one vibration period, Hamilton actions can be expressed as:

$$I = \int_0^{T_1} (T - V)dt = \int_0^{2\pi} (T - V)d\varphi. \quad (17)$$

According to Eq. (14),  $\dot{x}$ ,  $\dot{y}$ ,  $\dot{\psi}$ ,  $\ddot{x}$ ,  $\ddot{y}$  and  $\ddot{\psi}$  can be deduced. According to Eq. (2),  $\dot{x}_1''$ ,  $\dot{y}_1''$ ,  $\dot{x}_2''$  and  $\dot{y}_2''$  are obtained. Substitute them to the kinetic energy Eq. (3) and potential energy Eq. (5). Compared with  $\beta$ ,  $\varphi_1$  and  $\varphi_2$ ,  $\psi$  is so small that it could be neglected. Do an integral over the kinetic energy and potential energy within  $0 \sim 2\pi$  and substitute the integral value into Eq. (17). Then Eq. (17) can be expressed as follows:

$$\begin{aligned} I &\approx \int_0^{2\pi} (T - V)d\varphi = \frac{\pi}{2} [a_x^2(M_a\bar{\omega}^2 - k_{xy}) + a_y^2(M_a\bar{\omega}^2 - k_{xy}) + a_\psi^2(J\bar{\omega}^2 - k_\psi)] \\ &= \frac{\pi}{2} \left\{ \frac{(m_1r_1 + m_2r_2)^2\bar{\omega}^4(\bar{\omega}^2 - \omega_x^2)\sin^2\alpha}{M_a[(\omega_x^2 - \bar{\omega}^2)^2 + (2\xi_x\omega_x\bar{\omega})^2]} + \frac{(m_1r_1 + m_2r_2)^2\bar{\omega}^4(\bar{\omega}^2 - \omega_y^2)\cos^2\alpha}{M_a[(\omega_{ey}^2 - \bar{\omega}^2)^2 + (2\delta_{ey}\bar{\omega})^2]} \right. \\ &\quad \left. + \frac{(m_1r_1 + m_2r_2)^2l_0^2\bar{\omega}^4(\bar{\omega}^2 - \omega_\psi^2)\sin^2\alpha}{J_a[(\omega_\psi^2 - \bar{\omega}^2)^2 + (2\xi_\psi\omega_\psi\bar{\omega})^2]} \right\}. \end{aligned} \quad (18)$$

Expand  $\cos^2\alpha$  and  $\sin^2\alpha$  in Eq. (18) and merger the similar items which contain  $\sin 2\alpha$ ,  $\cos 2\alpha$  and constant terms. Then Eq. (18) can be rewritten as:

$$I \approx \frac{\pi}{2} (E\cos 2\alpha + C), \quad (19)$$

where:

$$\begin{aligned} E &= \frac{1}{2} \left\{ -\frac{(m_1r_1 + m_2r_2)^2\bar{\omega}^4(\bar{\omega}^2 - \omega_x^2)}{M_a[(\omega_x^2 - \bar{\omega}^2)^2 + (2\xi_x\omega_x\bar{\omega})^2]} + \frac{(m_1r_1 + m_2r_2)^2\bar{\omega}^4(\bar{\omega}^2 - \omega_y^2)}{M_a[(\omega_{ey}^2 - \bar{\omega}^2)^2 + (2\delta_{ey}\bar{\omega})^2]} \right. \\ &\quad \left. - \frac{(m_1r_1 + m_2r_2)^2\bar{\omega}^4l_0^2(\bar{\omega}^2 - \omega_\psi^2)}{J_a[(\omega_\psi^2 - \bar{\omega}^2)^2 + (2\xi_\psi\omega_\psi\bar{\omega})^2]} \right\}, \\ C &= \frac{1}{2} \left\{ \frac{(m_1r_1 + m_2r_2)^2\bar{\omega}^4(\bar{\omega}^2 - \omega_x^2)}{M_a[(\omega_x^2 - \bar{\omega}^2)^2 + (2\xi_x\omega_x\bar{\omega})^2]} + \frac{(m_1r_1 + m_2r_2)^2\bar{\omega}^4(\bar{\omega}^2 - \omega_y^2)}{M_a[(\omega_{ey}^2 - \bar{\omega}^2)^2 + (2\delta_{ey}\bar{\omega})^2]} \right. \\ &\quad \left. + \frac{(m_1r_1 + m_2r_2)^2l_0^2\bar{\omega}^4(\bar{\omega}^2 - \omega_\psi^2)}{J_a[(\omega_\psi^2 - \bar{\omega}^2)^2 + (2\xi_\psi\omega_\psi\bar{\omega})^2]} \right\}. \end{aligned} \quad (20)$$

The Hamilton's principle is described as Eq. (21):

$$\delta I + \frac{1}{\omega_0} \int_0^{2\pi} \sum_{i=1}^2 (F_i \delta q_i) d(\omega t) = 0. \quad (21)$$

Substitute (19) into (21) and arrange to obtain:

$$\sin 2\alpha = \frac{2(\Delta T_e - \Delta T_{fd})}{E} = \frac{\Delta T_e - \Delta T_{fd}}{H} = \frac{1}{D}. \quad (22)$$

In Eq. (22), define  $H$ ,  $D$  and  $W$  as the frequency capture moment, the self-synchronization coefficient and the stability coefficient. They can be expressed as follows:

$$\begin{aligned} H &= \frac{1}{2}E = \frac{(m_1 r_1 + m_2 r_2)^2 \bar{\omega}^2}{4} W, \\ D &= \frac{H}{\Delta T_e - \Delta T_{fd}} = \frac{\frac{1}{4}(m_1 r_1 + m_2 r_2)^2 \bar{\omega}^2 W}{(T_{e1} - T_{e2}) - (T_{fd1} - T_{fd2})}, \\ W &= -\frac{\bar{\omega}^2(\bar{\omega}^2 - \omega_x^2)}{M_a[(\omega_x^2 - \bar{\omega}^2)^2 + (2\xi_x \omega_x \bar{\omega})^2]} + \frac{\bar{\omega}^2(\bar{\omega}^2 - \omega_y^2)}{M_a[(\omega_{ey}^2 - \bar{\omega}^2)^2 + (2\delta_{ey} \bar{\omega})^2]} \\ &\quad - \frac{\bar{\omega}^2 l_0^2 (\bar{\omega}^2 - \omega_\psi^2)}{J_a[(\omega_\psi^2 - \bar{\omega}^2)^2 + (2\xi_\psi \omega_\psi \bar{\omega})^2]}. \end{aligned} \quad (23)$$

According to the above deduction, two eccentric rotors should have a constant phase difference in order to implement self-synchronous operation. Then we have Eq. (24):

$$|D| \geq 1 \quad \text{or} \quad \left| \frac{\Delta T_e - \Delta T_{fd}}{H} \right| \leq 1. \quad (24)$$

Substitute Eq. (23) into Eq. (24) and the condition of implementing synchronization can be deduced, which is that the frequency capture torque of two motors is greater than or equal to the torque difference between the residual electromagnetic torques of two motors.

When  $|D| > 1$ ,  $\arcsin(1/D)$  satisfied the following two ranges:

(1) When  $W/(\Delta T_e - \Delta T_{fd}) < 0$ , i.e.,  $D < -1$ ,  $\arcsin(1/D)$ , i.e., the phase difference  $2\alpha$  is in  $(-\pi, 0)$  range.

(2) When  $W/(\Delta T_e - \Delta T_{fd}) > 0$ , i.e.,  $D > 1$ ,  $\arcsin(1/D)$ , i.e., the phase difference  $2\alpha$  is in  $(0, \pi)$  range.

#### 4. The stability of self-synchronization motion

In the last two formula of Eq. (8), the swinging angular velocity  $\dot{\psi}$  is so small that it can be able to be neglected. Compared with  $\varphi_1$  and  $\varphi_2$ ,  $\psi$  is very small and can be ignored. Then the last two formula of Eq. (8) can be simplified as:

$$\begin{aligned} (J_{01} + m_1 r_1^2) \ddot{\varphi}_1 + f_{d1} \dot{\varphi}_1 &= T_{e1} - m_1 r_1 [\dot{y} \cos \varphi_1 - \ddot{x} \sin \varphi_1 + \ddot{\psi} (r_1 - l_0 \cos(\varphi_1 + \beta))], \\ (J_{02} + m_2 r_1^2) \ddot{\varphi}_2 + f_{d2} \dot{\varphi}_2 &= T_{e2} - m_2 r_2 [\dot{y} \cos \varphi_2 + \ddot{x} \sin \varphi_2 - \ddot{\psi} (r_2 - l_0 \cos(\varphi_2 + \beta))]. \end{aligned} \quad (25)$$

We assume:

$$\begin{aligned} \varphi_1 &= x_1, \quad \dot{\varphi}_1 = x_2, \quad \varphi_2 = x_3, \quad \dot{\varphi}_2 = x_4, \\ e_1 &= x_1 - x_3, \quad e_2 = x_2 - x_4, \\ m_1 &= m_2, \quad r_1 = r_2 = r, \quad J_{01} = J_{02}, \quad f_{d1} = f_{d2}. \end{aligned} \tag{26}$$

Then we can obtain:

$$\left\{ \begin{aligned} \dot{e}_1 &= e_2, \\ \dot{e}_2 &= \left\{ \begin{aligned} &\frac{1}{J_{01} + m_1 r^2} (T_{e1} - T_{e2}) - \frac{f_{d1}}{J_{01} + m_1 r^2} e_2 - \\ &-\frac{m_1 r}{J_{01} + m_1 r^2} [(\dot{y} - \ddot{\psi} l_0 \cos \beta) \cos x_1 - (\dot{y} + \ddot{\psi} l_0 \cos \beta) \cos x_3] + \\ &+\frac{m_1 r}{J_{01} + m_1 r^2} [(\dot{x} - \ddot{\psi} l_0 \sin \beta) \sin x_1 + (\dot{x} - \ddot{\psi} l_0 \sin \beta) \sin x_3] - \frac{2m_1 r^2}{J_{01} + m_1 r^2} \ddot{\psi} \end{aligned} \right\}. \end{aligned} \right. \tag{27}$$

According to Eq. (14) and Eq. (15), we can obtain  $\dot{x}$ ,  $\dot{y}$  and  $\ddot{\psi}$ . Substitute them into Eq. (27) and neglect the higher order terms, then we have:

$$\left\{ \begin{aligned} \dot{e}_1 &= e_2, \\ \dot{e}_2 &= \frac{T_{e1} - T_{e2}}{J_{01} + m_1 r^2} - \frac{f_{d1}}{J_{01} + m_1 r^2} e_2 - \frac{H'}{J_{01} + m_1 r^2} \text{sine}_1, \end{aligned} \right. \tag{28}$$

where:

$$\begin{aligned} H' &= \frac{m_1^2 r^2 \bar{\omega}^4 \cos \vartheta_y}{M_a \sqrt{(\omega_{ey}^2 - \bar{\omega}^2)^2 + (2\delta_{ey} \bar{\omega})^2}} - \frac{m_1^2 r^2 \bar{\omega}^4 \cos \vartheta_x}{M_a \sqrt{(\omega_x^2 - \bar{\omega}^2)^2 + (2\xi_x \omega_x \bar{\omega})^2}} \\ &- \frac{m_1^2 r^2 \bar{\omega}^4 l_0^2 \cos \vartheta_\psi}{J_a \sqrt{(\omega_\psi^2 - \bar{\omega}^2)^2 + (2\xi_\psi \omega_\psi \bar{\omega})^2}} \end{aligned} \tag{29}$$

The corresponding equilibrium point equation with Eq. (28) can be expressed as Eq. (30):

$$\left\{ \begin{aligned} e_2 &= 0, \\ \frac{1}{J_{01} + m_1 r^2} (T_{e1} - T_{e2}) - \frac{f_{d1}}{J_{01} + m_1 r^2} e_2 - \frac{H'}{J_{01} + m_1 r^2} \text{sine}_1 &= 0. \end{aligned} \right. \tag{30}$$

According to Eq. (30), when:

$$\text{sine}_1 = \left| \frac{\Delta T_e}{H'} \right| \leq 1. \tag{31}$$

The equilibrium point of the vibration system is obtained as  $(\arcsin \Delta T_e / H', 0)$ . As can be seen from the equilibrium point and Eq. (26), when the system is stable at this point, it can implement self-synchronization operation of zero angular velocity difference. The phase plane singularity at the equilibrium point of Eq. (30) is analyzed and the characteristic equation of Eq. (30) is obtained as the following:

$$\lambda^2 + \frac{f_{d1}}{J_{01} + m_1 r^2} \lambda + \frac{H'}{J_{01} + m_1 r^2} \text{cose}_1 = 0, \tag{32}$$

where  $e_1 = \arcsin \Delta T_e / H'$  is the phase difference of two motors at the equilibrium point of the system. Based on the once approximate method of stability of nonlinear system, when:

$$-\arccos \left[ \frac{f_{d1}^2}{4H'(J_{01} + m_1 r^2)} \right] < e_1 < \arccos \left[ \frac{f_{d1}^2}{4H'(J_{01} + m_1 r^2)} \right]. \quad (33)$$

The vibrating system have asymptotically stable focus on the phase plane and the focus is the equilibrium point  $(\arcsin \Delta T_e / H', 0)$  of nonlinear vibrating system at the time of non-resonant.

Substitute the phase difference  $e_1 = \arcsin \Delta T_e / H'$  of two motors at the equilibrium point of the system to Eq. (33) and neglect the higher order term. The condition of gradual stability when the vibration system operate synchronously at equilibrium point can be deduced:

$$|H'| > \sqrt{(T_{e1} - T_{e2})^2 + \frac{f_{d1}^4}{16(J_{01} + m_1 r^2)^2}}. \quad (34)$$

## 5. Conclusions

A single mass vibration system driven by two counter-rotating motors is studied in this paper. The hard characteristics of vibration springs in the vertical direction is considered and based on the Lagrange Equations, the mechanic-electric coupling dynamic equation of the vibration system is established. When the system is working in the non-resonant case, the meaningful solutions for project in horizon, vertical and swing directions are obtained by gradual method of nonlinear system.

By the Hamilton theory, the condition of the self-synchronization implementation of the system was obtained, which is that the frequency capture moment  $H$  of two motors is greater than the absolute value of the output electromagnetic torque difference, i.e.,  $|\Delta T_e - \Delta T_{fd}|$ . According to Eq. (24), when  $|D| > 1$ ,  $\arcsin(1/D)$  satisfied the following two ranges: when  $W/(\Delta T_e - \Delta T_{fd}) < 0$ , i.e.,  $D < -1$ ,  $\arcsin(1/D)$ , i.e., the phase difference  $2\alpha$  is in  $(-\pi, 0)$  range. When  $W/(\Delta T_e - \Delta T_{fd}) > 0$ , i.e.,  $D > 1$ ,  $\arcsin(1/D)$ , i.e., the phase difference  $2\alpha$  is in  $(0, \pi)$  range.

Applying once approximate method of stability of nonlinear system, the asymptotic stability condition of the vibration system at equilibrium point are deduced, which can be expressed as the following:

$$|H'| > \sqrt{(T_{e1} - T_{e2})^2 + f_{d1}^4 / 16(J_{01} + m_1 r^2)^2}.$$

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