

1416. Bending of inhomogeneous sandwich plates with viscoelastic cores

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Abstract. This article presents the bending analysis of an inhomogeneous composite sandwich rectangular plate with viscoelastic core. The sinusoidal plate theory as well as other familiar shear deformation plate theories is used. Different types of intermediate plates are considered according to the thickness of all layers and the symmetry of the plate. Illyushin's approximation and the effective moduli methods are used to treat the governing equations of sandwich plates that reinforced with inhomogeneous fibers. Various results for deflections of and some stresses in such plates are presented. A comparison study is made to investigate the effect of time, inhomogeneity, and constitutive parameters as well as the effect of span-to-thickness and aspect ratios on the bending response of the sandwich plates.

Keywords: sandwich plates, viscoelastic cores, inhomogeneous, sinusoidal plate theory.

1. Introduction

Sandwich structure is widely used in modern applications, especially in civil, marine and mechanical engineering industries due to its light weight and high rigidity. Sandwich plates are basically a special form of fiber-reinforced rectangular plates composed of alternative arrangement of thin stiff layers and thick flexible cores. They also made up of two thin strong, stiff face layers which resist bending bonded to a relatively thicker, less dense core layer to resist shear force. There has recently been a great interest to study the behavior of sandwich structures subjected to static or dynamic loads [1-4]. Whitney [5] and Vinson [6] have discussed the structure of sandwich plates and emphasized the importance of including shear flexibility of their cores. Pagano [7] and Pagano and Hatfield [8] have presented exact 3-D elasticity solutions for the stresses in the composite laminates and sandwich plates. In fact, Zenkour [9] has presented recent results more general than the well-known of Pagano [7] and Pagano and Hatfield [8] and they served as benchmark solutions for other researchers. Zenkour and his colleagues [10-15] have presented a 2-D solution for the bending and/or free vibration of functionally graded (FG) sandwich rectangular plates. They have investigated the thermo-mechanical bending analysis of FG sandwich plates. In addition, they have presented much elastic foundation analyses of uniformly loaded FG viscoelastic sandwich plates.

The classical plate theory (CPT) gives inaccurate results for thick or may be moderately thick rectangular plates because it ignores transverse shear effects. However, it provides reasonable results for thin plates. The first-order shear deformation plate theory (FPT) [16] is an improvement on the CPT. It accounts for transverse shear effects but needs a shear correction factor to give reasonable results. Higher-order shear deformation plate theory (HPT) [17] can represent the kinematics behavior of most laminated plates better. It does not require shear correction factors and can be used to compute inter-laminar stresses more accurately.

In this article, the sinusoidal shear deformation theory of plate (SPT) [18] as well as other

familiar plate theories (CPT, FPT and HPT) is used to study the static bending of a viscoelastic rectangular sandwich plate. The faces of this sandwich plate made of the same elastic isotropic material while its core is made of a viscoelastic one. With the help of Illyushin approximation method [19], and the effective-moduli method [20], a wide range of deflections and stresses is presented to the bending of sandwich plates. A comparison between the various theories of plates is investigated. Additional results for the bending of fiber-reinforced inhomogeneous viscoelastic sandwich plates are added to serve as benchmarks for future comparisons with other researchers.

2. Basic equations

Let us consider a symmetric sandwich plate of uniform thickness h , length a , and width b and composed of three inhomogeneous layers (see Fig. 1). To describe infinitesimal deformations of the three-layer sandwich plate one can use the Cartesian coordinates (x, y, z) . The plane $z = 0$ represents the mid-plane of the plate while $z = \pm h/2$ represent its external bounding planes. The upper surface is subjected to a normal traction $\sigma_z = -q(x, y)$, while its lower surface is traction free. Different types of symmetric sandwich plate are considered according to the thickness of the core and faces. For example, the (1-5-1) sandwich plate is given when the core thickness of the plate is equal to five times the face thickness, i.e., $h_1 = -5h/14$ and $h_2 = +5h/14$. In the (2-1-2) sandwich plate, the core thickness is half the face thickness, i.e., $h_1 = -h/10$ and $h_2 = +h/10$. The (1-1-1) sandwich plate is made of three layers with equal thickness, i.e., $h_1 = -h/6$ and $h_2 = +h/6$. Also, the coreless sandwich plate (1-0-1) is considered with two face layers of equal thickness, i.e., $h_1 = h_2 = 0$.

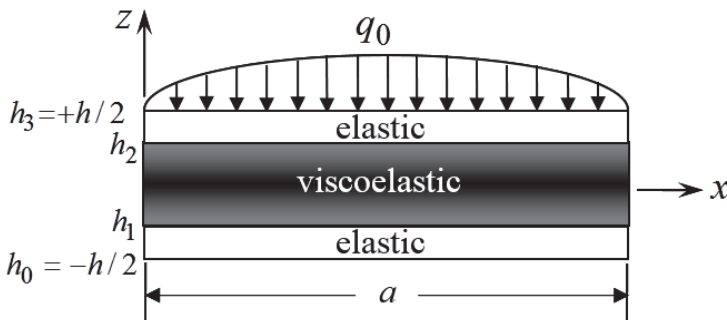


Fig. 1. Coordinates and geometry for the viscoelastic sandwich plate

The material properties of the inhomogeneous layers are smoothly varying in the thickness z -direction only. That is:

$$c_{ij}^{(k)}(z) = \bar{c}_{ij}^{(k)} e^{-nz/h}, \quad k = 1, 2, 3, \quad (1)$$

where n is the inhomogeneity parameter and $\bar{c}_{ij}^{(k)}$ are the transformation elastic coefficients that depending on the material properties of the homogeneous k th layer:

$$\begin{aligned} \bar{c}_{ii}^{(k)} &= \frac{E_k}{1 - \nu_k^2}, & \bar{c}_{12}^{(k)} &= \frac{\nu_k E_k}{1 - \nu_k^2}, \\ \bar{c}_{jj}^{(k)} &= \frac{E_k}{2(1 + \nu_k)}, & i &= 1, 2, \quad j = 4, 5, 6, \end{aligned} \quad (2)$$

in which E_k is Young's modulus of layer k and ν_k is its Poisson's ratio. The displacement field of a point at (x, y, z) in the plate may be written as [18]:

$$\begin{aligned} V_1 &= u_1 - z \frac{\partial u_3}{\partial x} + \psi(z)u_5, \\ V_2 &= u_2 - z \frac{\partial u_3}{\partial y} + \psi(z)u_4, \\ V_3 &= u_3, \end{aligned} \tag{3}$$

where (V_1, V_2, V_3) are functions of the spatial co-ordinate that represent the displacements corresponding to the co-ordinate system; (u_1, u_2, u_3) are the displacements along the axes x , y and z , respectively, and u_4 and u_5 are the rotations about the x - and y -axes. The generalized displacements u_i are functions of x and y . The shape function $\psi(z)$ may be chosen such that:

$$\left. \frac{d\psi}{dz} \right|_{z=\pm h/2} = 0 \quad \text{or/and} \quad \int_{-h/2}^{+h/2} \psi(z) dz = 0. \tag{4}$$

The first condition in Eq. (4) means that the transverse shear strain vanishes on the bounding planes $z = \pm h/2$. The shear deformation theory that satisfies this condition dose not requires any shear-correction factors. In general, the second condition in Eq. (4) may be satisfied for most two-dimensional theories. In more details, there is no shape function for CPT, i.e. $\psi(z) = 0$ while $\psi(z) = z$ for FPT, $\psi(z) = z[1 - \frac{4}{3}(\frac{z}{h})^2]$ for HPT, and $\psi(z) = h/\pi \sin(\pi z/h)$ for SPT [20].

The strain-displacement relations are given by:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{Bmatrix} + \psi(z) \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_6 \end{Bmatrix}, \quad \varepsilon_3 = 0, \quad \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} = \frac{d\psi}{dz} \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \end{Bmatrix}, \tag{5}$$

where:

$$\begin{aligned} \varepsilon_1^0 &= \frac{\partial u_1}{\partial x}, \quad \varepsilon_2^0 = \frac{\partial u_2}{\partial y}, \quad \varepsilon_4^0 = u_4, \quad \varepsilon_5^0 = u_5, \quad \varepsilon_6^0 = \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}, \\ \kappa_1 &= -\frac{\partial^2 u_3}{\partial x^2}, \quad \kappa_2 = -\frac{\partial^2 u_3}{\partial y^2}, \quad \kappa_6 = -2\frac{\partial^2 u_3}{\partial x \partial y}, \\ \eta_1 &= \frac{\partial u_5}{\partial x}, \quad \eta_2 = \frac{\partial u_4}{\partial y}, \quad \eta_6 = \frac{\partial u_4}{\partial x} + \frac{\partial u_5}{\partial y}. \end{aligned} \tag{6}$$

Each layer of the present sandwich plate may be treated as an individual inhomogeneous plate. So, the stress-strain relations for the k th layer, according to the inclusion of the transverse shear deformation, can be expressed as:

$$\begin{Bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \end{Bmatrix}^{(k)} = \frac{E_k e^{-nz/h}}{1 - \nu_k^2} \begin{bmatrix} 1 & \nu_k \\ \nu_k & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{Bmatrix}, \quad \{\bar{\sigma}_4, \bar{\sigma}_5, \bar{\sigma}_6\}^{(k)} = \frac{E_k e^{-nz/h}}{2(1 + \nu_k)} \{\varepsilon_4, \varepsilon_5, \varepsilon_6\}. \tag{7}$$

The principle of virtual displacement may be expressed for this case as:

$$\int_{-h/2}^{+h/2} \int_{\Omega} [\bar{\sigma}_1^{(k)} \delta \varepsilon_1 + \bar{\sigma}_2^{(k)} \delta \varepsilon_2 + \dots] d\Omega dz - \int_{\Omega} q \delta u_3 d\Omega = 0, \tag{8}$$

or:

$$\int_{\Omega} [N_1 \delta \varepsilon_1^0 + N_2 \delta \varepsilon_2^0 + N_6 \delta \varepsilon_6^0 + M_1 \delta \kappa_1 + M_2 \delta \kappa_2 + M_6 \delta \kappa_6 + Q_4 \delta \varepsilon_4^0 + Q_5 \delta \varepsilon_5^0 + S_1 \delta \eta_1 + S_2 \delta \eta_2 + S_6 \delta \eta_6 - q \delta u_3] d\Omega = 0, \quad (9)$$

where N_i are the stress resultants, M_i are the stress couples, Q_j are the transverse shear stress resultants, and S_i are stress couples added to associate with the transverse shear effects:

$$\{N_i, M_i, S_i\} = \sum_{k=1}^3 \int_{h_{k-1}}^{h_k} \bar{\sigma}_i^{(k)} \{1, z, \psi(z)\} dz, \quad (10)$$

$$Q_j = \sum_{k=1}^3 \int_{h_{k-1}}^{h_k} \bar{\sigma}_j^{(k)} \frac{d\psi}{dz} dz, \quad i = 1, 2, \quad j = 4, 5,$$

in which h_{k-1} and h_k are the bottom and top z -coordinates of the layer k .

3. Governing equations

Eq. (8) is used to obtain the governing equilibrium equations by integrating the displacement gradient in ε_i by parts and setting the coefficients of δu_i to zero separately. In this case, we get:

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = 0, \quad \frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = 0,$$

$$\frac{\partial^2 M_1}{\partial x^2} + 2 \frac{\partial^2 M_6}{\partial x \partial y} + \frac{\partial^2 M_2}{\partial y^2} + q = 0, \quad (11)$$

$$\frac{\partial S_1}{\partial x} + \frac{\partial S_6}{\partial y} - Q_5 = 0, \quad \frac{\partial S_6}{\partial x} + \frac{\partial S_2}{\partial y} - Q_4 = 0.$$

Substituting Eq. (7) into Eq. (10), the resultants N_i , M_i , Q_j and S_i can be represented in terms of the total strains:

$$\begin{Bmatrix} N \\ M \\ S \end{Bmatrix} = \begin{bmatrix} A & B & D \\ B & F & H \\ D & H & L \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \kappa \\ \eta \end{Bmatrix}, \quad Q = G\gamma, \quad (12)$$

where:

$$N = \begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix}, \quad M = \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix}, \quad S = \begin{Bmatrix} S_1 \\ S_2 \\ S_6 \end{Bmatrix}, \quad \varepsilon = \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{Bmatrix}, \quad \kappa = \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{Bmatrix}, \quad \eta = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_6 \end{Bmatrix}, \quad (13a)$$

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}, \quad (13b)$$

$$F = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{12} & F_{22} & 0 \\ 0 & 0 & F_{66} \end{bmatrix}, \quad H = \begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{12} & H_{22} & 0 \\ 0 & 0 & H_{66} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & L_{12} & 0 \\ L_{12} & L_{22} & 0 \\ 0 & 0 & L_{66} \end{bmatrix}, \quad (13c)$$

$$Q = \begin{Bmatrix} Q_4 \\ Q_5 \end{Bmatrix}, \quad \gamma = \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \end{Bmatrix}, \quad G = \begin{bmatrix} G_{44} & 0 \\ 0 & G_{55} \end{bmatrix}. \quad (13d)$$

The stiffness coefficients A_{ij} , B_{ij} , ... etc., are defined as:

$$\begin{aligned} \{A_{ij}, B_{ij}, D_{ij}\} &= \sum_{k=1}^3 \int_{h_{k-1}}^{h_k} c_{ij}^{(k)} \{1, z, \psi(z)\} dz, \\ \{F_{ij}, H_{ij}, L_{ij}\} &= \sum_{k=1}^3 \int_{h_{k-1}}^{h_k} c_{ij}^{(k)} \{z^2, z\psi(z), [\psi(z)]^2\} dz, \\ G_{il} &= \sum_{k=1}^3 \int_{h_{k-1}}^{h_k} K_l c_{il}^{(k)} \left[\frac{d\psi}{dz} \right]^2 dz, \quad i, j = 1, 2, 6, \quad l = 4, 5, \end{aligned} \quad (14)$$

where K_4 and K_5 represent the shear correction factors for FPT only. Their suitable values for FPT are $K_4 = K_5 = 5/6$. Otherwise, they should be unity.

4. Closed-form solution

The closed-form solution of Eq. (11) for the sandwich plate may be obtained here. The simply-supported boundary conditions at the side edges are imposed as follows:

$$\begin{aligned} u_2 = u_3 = u_4 = N_1 = M_1 = S_1 = 0 \quad \text{at } x = 0, a, \\ u_1 = u_3 = u_5 = N_2 = M_2 = S_2 = 0 \quad \text{at } y = 0, b. \end{aligned} \quad (15)$$

To solve the present problem, the external force is represented according to Navier-type in the form of a sinusoidally distributed load as:

$$q(x, y) = q_0 \sin(\lambda x) \sin(\mu y), \quad (16)$$

where $\lambda = \pi/a$ and $\mu = \pi/b$. The plate-center intensity of the load is denoted by q_0 . In addition, one can assume the following solution forms for u_i that satisfies the boundary conditions:

$$\begin{Bmatrix} (u_1, u_5) \\ u_3, \\ (u_2, u_4) \end{Bmatrix} = \begin{Bmatrix} (U_1, U_5) \cos(\lambda x) \sin(\mu y) \\ U_3 \sin(\lambda x) \sin(\mu y) \\ (U_2, U_4) \sin(\lambda x) \cos(\mu y) \end{Bmatrix}, \quad (17)$$

where U_i are arbitrary parameters to be determined. Substituting Eqs. (16) and (17) into Eq. (11), one obtains:

$$[C]\{\Delta\} = \{F\}, \quad (18)$$

where $\{\Delta\}$ and $\{F\}$ denote the columns:

$$\begin{aligned} \{\Delta\}^T &= \{U_1, U_2, U_3, U_4, U_5\}, \\ \{F\}^T &= \{0, 0, -q_0, 0, 0\}, \end{aligned} \quad (19)$$

where “ T ” denotes the vector transpose of $\{F\}$ and $\{\Delta\}$. The elements $C_{ij} = C_{ji}$ of the coefficient matrix $[C]$ are given by:

$$\begin{aligned} C_{11} &= -\lambda^2 A_{11} - \mu^2 A_{66}, & C_{12} &= -\lambda\mu(A_{12} + A_{66}), & C_{13} &= \lambda[\lambda^2 B_{11} + \mu^2(B_{12} + 2B_{66})], \\ C_{14} &= -\lambda^2 D_{11} - \mu^2 D_{66}, & C_{15} = C_{24} &= -\lambda\mu(D_{12} + D_{66}), & C_{22} &= -\lambda^2 A_{66} - \mu^2 A_{22}, \\ C_{23} &= \mu[\lambda^2(B_{12} + 2B_{66}) + \mu^2 B_{22}], & C_{25} &= -\lambda^2 D_{66} - \mu^2 D_{22}, \\ C_{33} &= -\lambda^4 F_{11} - 2\lambda^2 \mu^2(F_{12} + 2F_{66}) - \mu^4 F_{22}, & C_{34} &= \lambda[\lambda^2 H_{11} + \mu^2(H_{12} + 2H_{66})], \\ C_{35} &= \mu[\lambda^2(H_{12} + 2H_{66}) + \mu^2 H_{22}], & C_{44} &= -\lambda^2 L_{11} - \mu^2 L_{66} - G_{55}, \\ C_{45} &= -\lambda\mu(L_{12} + L_{66}), & C_{55} &= -\lambda^2 L_{66} - \mu^2 L_{22} - G_{44}. \end{aligned} \quad (20)$$

The stress components may be obtained by substituting Eq. (17) into Eq. (6). They are

appeared in terms of the arbitrary parameters U_i and the material properties as:

$$\begin{aligned}
 \bar{\sigma}_1^{(k)} &= -\frac{E_k e^{-\frac{nz}{h}}}{1 - \nu_k^2} \left[\lambda U_1 + \nu_k \mu U_2 - z(\lambda^2 + \nu_k \mu^2) U_3 \right] \sin(\lambda x) \sin(\mu y), \\
 \bar{\sigma}_2^{(k)} &= -\frac{E_k e^{-\frac{nz}{h}}}{1 - \nu_k^2} \left[\nu_k \lambda U_1 + \mu U_2 - z(\nu_k \lambda^2 + \mu^2) U_3 \right] \sin(\lambda x) \sin(\mu y), \\
 \bar{\sigma}_4^{(k)} &= \frac{E_k e^{-\frac{nz}{h}}}{2(1 + \nu_k)} \frac{d\psi}{dz} U_4 \sin(\lambda x) \cos(\mu y), \\
 \bar{\sigma}_5^{(k)} &= \frac{E_k e^{-\frac{nz}{h}}}{2(1 + \nu_k)} \frac{d\psi}{dz} U_5 \cos(\lambda x) \sin(\mu y), \\
 \bar{\sigma}_6^{(k)} &= \frac{E_k e^{-\frac{nz}{h}}}{2(1 + \nu_k)} [\mu U_1 + \lambda U_2 - 2z\lambda\mu U_3 + \psi(z)(\mu U_5 + \lambda U_4)] \cos(\lambda x) \cos(\mu y).
 \end{aligned} \tag{21}$$

5. Viscoelastic solution

In this problem, the core is made of a viscoelastic isotropic material while the face sheets are made of an isotropic material with $E_1 = E_3 = E$ and $\nu_1 = \nu_3 = \nu$. However, the viscoelastic properties are given by:

$$E_2 = \frac{9K\bar{\omega}}{2 + \bar{\omega}}, \quad \nu_2 = \frac{1 - \bar{\omega}}{2 + \bar{\omega}}, \tag{22}$$

where K is the bulk modulus (the coefficient of volume compression) and $\bar{\omega}$ is the dimensionless kernel of relaxation function. It is assumed that K is not relaxed, i.e. $K = \text{constant}$.

Firstly, the corresponding elastic problem is solved numerically for variable values of $\bar{\omega}$. Then, one can use the realization method of elastic solution for solving the problem of quasi-static linear viscoelastic composite. In the elastic case u_i are functions of x, y and $\bar{\omega}$ while in the viscoelastic case, they are operator functions of x, y and time t . According to Illyushin's approximation method (see Pobedrya [19]), the displacement functions u_i can be represented for layer k in the form:

$$u_i^{(k)}(x, y, \bar{\omega}) = \sum_{j=1}^5 A_{ij}^{(k)}(x, y) \Phi_j(\bar{\omega}), \quad i = 1, \dots, 5, \tag{23}$$

where $\Phi_j(\bar{\omega})$ are five known kernels. They constructed on the basis of $\bar{\omega}$ and one can choose them in the form:

$$\Phi_1 = 1, \quad \Phi_2 = \bar{\omega}, \quad \Phi_3 = \bar{\pi}, \quad \Phi_4 = \bar{g}_{\beta_1}, \quad \Phi_5 = \bar{g}_{\beta_2}, \tag{24}$$

where $\bar{g}_{\beta_m} = 1/1 + \beta_m \bar{\omega}$, $m = 1, 2$. The coefficients $A_{ij}^{(k)}(x, y)$ may be determined from the following algebraic equations system:

$$\sum_{j=1}^5 J_{ij} A_{ij}^{(k)}(x, y) = B_i^{(k)}(x, y), \quad i = 1, \dots, 5, \tag{25}$$

where:

$$J_{ij} = \int_0^1 \Phi_i(\bar{\omega})\Phi_j(\bar{\omega})d\bar{\omega}, \quad B_i^{(k)}(x, y) = \int_0^1 \Phi_i(\bar{\omega})u_i^{(k)}(x, y, \bar{\omega})d\bar{\omega}. \quad (26)$$

The viscoelastic solution can now record to get explicit unified form for the displacements $u_i^{(k)}(x, y, t)$ as:

$$u_i^{(k)}(x, y, t) = A_{i1}^{(k)}q_0(t) + A_{i2}^{(k)}\int_0^1 \omega(t - \tau)dq_0(t) + A_{i3}^{(k)}\int_0^1 \pi(t - \tau)dq_0(t) \\ + A_{i4}^{(k)}\int_0^1 g_{\beta_1}(t - \tau)dq_0(t) + A_{i5}^{(k)}\int_0^1 g_{\beta_2}(t - \tau)dq_0(t). \quad (27)$$

Taking $q_0(t) = \bar{q}_0H(t)$ where $H(t)$ is the Heaviside's unit step function:

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases} \quad (28)$$

then, Eq. (27) for the two problems takes the form:

$$u_i^{(k)}(x, y, t) = \bar{q}_0[A_{i1}^{(k)}H(t) + A_{i2}^{(k)}\omega(t) + A_{i3}^{(k)}\pi(t) + A_{i4}^{(k)}g_{\beta_1}(t) + A_{i5}^{(k)}g_{\beta_2}(t)], \quad (29)$$

where $\omega(t) \equiv \bar{\omega}$, $\pi(t) \equiv \bar{\pi}$, and $g_{\beta_m}(t) \equiv \bar{g}_{\beta_m}$.

Assuming an exponential relaxation function:

$$\omega(t) = c_1 + c_2e^{-\alpha t}, \quad (30)$$

where c_1 and c_2 are constants, $\alpha = 1/t_s$ and t_s is the relaxation time. The functions $\pi(t)$ and $g_{\beta_m}(t)$ can be determined by deducing their Laplace-Carson transforms from the known ones of $\omega(t)$ [21, 22]. They take the following forms:

$$\pi(t) = \frac{1 - c_3e^{-c_4t}}{c_1}, \quad g_{\beta_m}(t) = \frac{1 - c_5^m e^{-c_6^m t}}{1 + c_1\beta_m}, \quad m = 1, 2, \quad \tau = \alpha t, \quad (31)$$

where:

$$c_3 = \frac{c_2}{c_1 + c_2}, \quad c_4 = \frac{c_1}{c_1 + c_2}, \quad c_5^m = \frac{c_2\beta_m}{1 + (c_1 + c_2)\beta_m}, \quad c_6^m = \frac{1 + c_1\beta_m}{1 + (c_1 + c_2)\beta_m}. \quad (32)$$

So, the displacements are given in the final form, in terms of the time t or the time parameter τ , by:

$$u_i^{(k)}(x, y, t) = \bar{q}_0 \left\{ A_{i1}^{(k)}H(t) + A_{i2}^{(k)}(c_1 + c_2e^{-\tau}) + \frac{A_{i3}^{(k)}}{c_1}(1 - c_3e^{-c_4\tau}) \right. \\ \left. + \frac{A_{i4}^{(k)}}{1 + c_1\beta_1}(1 - c_5^1 e^{-c_6^1\tau}) + \frac{A_{i5}^{(k)}}{1 + c_1\beta_2}(1 - c_5^2 e^{-c_6^2\tau}) \right\}. \quad (33)$$

Once again, the same technique is used to obtain the stresses by replacing only $u_i^{(k)}(x, y, t)$ with $\bar{\sigma}_i^{(k)}(x, y, z, t)$. So, the bending stresses in terms of the time t or the time parameter τ are given by:

$$\bar{\sigma}_i^{(k)}(x, y, z, t) = \bar{q}_0 \left\{ \bar{A}_{i1}^{(k)} H(t) + \bar{A}_{i2}^{(k)} (c_1 + c_2 e^{-\tau}) + \frac{\bar{A}_{i3}^{(k)}}{c_1} (1 - c_3 e^{-c_4 \tau}) + \frac{\bar{A}_{i4}^{(k)}}{1 + c_1 \beta_1} (1 - c_5^1 e^{-c_6^1 \tau}) + \frac{\bar{A}_{i5}^{(k)}}{1 + c_1 \beta_2} (1 - c_5^2 e^{-c_6^2 \tau}) \right\} \quad (34)$$

6. Numerical examples and discussions

Numerical examples are obtained for the deflections of and stresses in viscoelastic sandwich plates. The relaxation time α is still unknown and the time parameter $\tau = \alpha t$ is given in terms of it. The constitutive parameter is given by $\zeta = E/K$. In addition, it is assumed, unless otherwise stated, that $\nu = 0.3$, $b = 3a$, $\zeta = 0.2$, $a = 10h$, $n = 4$, $c_1 = 0.1$, and $c_2 = 0.9$.

The following dimensionless quantities are used throughout the calculations ($Z = z/h$):

$$\{\sigma_1, \sigma_2\} = \frac{h}{a\bar{q}_0} \{\bar{\sigma}_1, \bar{\sigma}_2\} \left(\frac{a}{2}, \frac{b}{2}, Z \right), \quad \sigma_6 = \frac{h}{a\bar{q}_0} \bar{\sigma}_6(0, 0, Z),$$

$$\sigma_4 = \bar{\sigma}_4 \left(\frac{a}{2}, 0, Z \right), \quad \sigma_5 = \bar{\sigma}_5 \left(0, \frac{b}{2}, Z \right), \quad w = \frac{Kh^2}{a^3 \bar{q}_0} u_3 \left(\frac{a}{2}, \frac{b}{2} \right). \quad (35)$$

Table 1 contains the dimensionless deflections w of inhomogeneous (1-5-1) viscoelastic sandwich plates. The effect of the side-to-thickness ratio a/h , the constitutive parameter ζ , and the aspect ratio b/a is presented. The results show that the dimensionless deflection w increases as the side-to-thickness ratio a/h and aspect ratio b/a increase while it decreases as the constitutive parameter ζ increases. The HPT yields results closer than FPT to those obtained using SPT. The CPT failed to give reliable results for thicker plates ($a/h = 5$).

Table 1. Effects of the constitutive parameter ζ , side-to-thickness ratio a/h , and aspect ratio b/a on the dimensionless deflection w of the (1-5-1) sandwich plate

$\frac{a}{h}$	$\frac{b}{h}$	$\zeta = 0.1$				$\zeta = 0.5$			
		SPT	HPT	FPT	CPT	SPT	HPT	FPT	CPT
5	0.5	0.11835	0.11810	0.11352	0.09093	0.06963	0.06945	0.06175	0.04856
	1.0	0.63719	0.63650	0.62479	0.56831	0.35665	0.35609	0.33647	0.30348
	1.5	1.18499	1.18403	1.16774	1.08955	0.65556	0.65476	0.62749	0.58181
	2.0	1.56521	1.56409	1.54524	1.45488	0.86217	0.86123	0.82968	0.77690
10	0.5	0.19565	0.19551	0.19315	0.18186	0.10777	0.10765	0.10371	0.09711
	1.0	1.17115	1.17079	1.16486	1.13663	0.63366	0.63336	0.62345	0.60695
	1.5	2.22690	2.22641	2.21819	2.17910	1.20062	1.20020	1.18646	1.16363
	2.0	2.96501	2.96444	2.95495	2.90977	1.59656	1.59606	1.58019	1.55380
20	0.5	0.37063	0.37055	0.36937	0.36372	0.19957	0.19951	0.19752	0.19423
	1.0	2.29053	2.29034	2.28737	2.27326	1.22728	1.22712	1.22215	1.21391
	1.5	4.38211	4.38186	4.37774	4.35820	2.34577	2.34555	2.33867	2.32725
	2.0	5.84717	5.84688	5.84212	5.81954	3.12899	3.12874	3.12080	3.10760

Tables 2-4 contain the stresses in the inhomogeneous (1-5-1) viscoelastic sandwich plates at different values of side-to-thickness ratio a/h , constitutive parameter ζ and aspect ratio b/a , respectively. The results given in Table 2 show that the stresses, except σ_6 , increase as the side-to-thickness ratio a/h increases. Table 3 shows that the stresses σ_1 and σ_6 increase with the increase of ζ while the rest of stresses are decreasing. Table 4 shows that the stresses increase with the increase of the aspect ratio b/a . In these figures, CPT gives similar results as FPT and may be ignore. Once again, HPT yields results closer than FPT to those obtained using SPT.

Figs. 2 and 3 illustrate the transverse shear stresses σ_4 and σ_5 through-the-thickness of the (1-5-1) sandwich plate using various theories. It can be seen that the shear stresses take larger

values at the core (viscoelastic) layer and the HPT yields results very close to those obtained using SPT. The FPT gives non-zero shear stresses at the top and bottom surfaces of the plate. Also, the SPT gives accurate results, especially transverse shear stresses, compared with other theories including HPT. This is an expected because SPT is the generalized one.

Table 2. Dimensionless stresses versus the side-to-thickness ratio a/h in the (1-5-1) sandwich plate

$\frac{a}{h}$	Theory	σ_1	σ_2	σ_6	σ_4	σ_5
5	SPT	0.31925	0.13877	0.31451	0.11605	0.34816
	HPT	0.31911	0.13892	0.31397	0.11124	0.33373
	FPT	0.31687	0.14008	0.30691	0.06777	0.20332
	CPT	0.31687	0.14008	0.30691	—	—
10	SPT	0.63494	0.13975	0.30881	0.11627	0.34881
	HPT	0.63487	0.13979	0.30868	0.11141	0.33422
	FPT	0.63374	0.14008	0.30691	0.06777	0.20332
	CPT	0.63374	0.14008	0.30691	—	—
20	SPT	1.26809	0.14000	0.30738	0.11632	0.34897
	HPT	1.26805	0.14001	0.30735	0.11145	0.33435
	FPT	1.26749	0.14008	0.30691	0.06777	0.20332
	CPT	1.26749	0.14008	0.30691	—	—
50	SPT	3.16896	0.14007	0.30698	0.11634	0.34901
	HPT	3.16895	0.14007	0.30698	0.11146	0.33438
	FPT	3.16872	0.14008	0.30691	0.06777	0.20332
	CPT	3.16872	0.14008	0.30691	—	—

Table 3. Dimensionless stresses versus the constitutive parameter ζ in the (1-5-1) sandwich plate

ζ	Theory	σ_1	σ_2	σ_6	σ_4	σ_5
0.1	SPT	0.38591	0.16619	0.21512	0.12355	0.37064
	HPT	0.38585	0.16622	0.21503	0.11850	0.35550
	FPT	0.38506	0.16646	0.21391	0.07983	0.23950
	CPT	0.38506	0.16646	0.21391	—	—
0.2	SPT	0.63494	0.13975	0.30881	0.11627	0.34881
	HPT	0.63487	0.13979	0.30868	0.11141	0.33422
	FPT	0.63374	0.14008	0.30691	0.06777	0.20332
	CPT	0.63374	0.14008	0.30691	—	—
0.3	SPT	0.83231	0.12403	0.36241	0.11143	0.33428
	HPT	0.83224	0.12407	0.36225	0.10651	0.31952
	FPT	0.83092	0.12439	0.35999	0.05888	0.17664
	CPT	0.83092	0.12439	0.35999	—	—
0.5	SPT	1.15235	0.10514	0.42327	0.10465	0.31395
	HPT	1.15228	0.10519	0.42305	0.09939	0.29818
	FPT	1.15069	0.10552	0.41999	0.04664	0.13991
	CPT	1.15069	0.10552	0.41999	—	—
0.9	SPT	1.64104	0.08498	0.48200	0.09541	0.28622
	HPT	1.64097	0.08502	0.48165	0.08939	0.26817
	FPT	1.63904	0.08535	0.47734	0.03294	0.09882
	CPT	1.63904	0.08535	0.47734	—	—

In Figs. 4 and 5, the SPT is used to plot the shear stresses σ_4 and σ_5 through-the-thickness of different kinds of the sandwich plates. It is seen that, both of σ_4 and σ_5 take different behaviors according to the thickness of the core layer compared with the thickness of the other faces. If there is no core as in the case of (1-0-1) plate, this means that the plate becomes fully elastic and the shear stresses take the same curve-related shape with maximum points near the lower faces. For other sandwich plates, the maximum values of the shear stresses occur at the first interface. The values of σ_5 may be three times those of σ_4 .

Table 4. Dimensionless stresses versus the aspect ratio b/a in the (1-5-1) sandwich plate

$\frac{b}{a}$	Theory	σ_1	σ_2	σ_6	σ_4	σ_5
0.1	SPT	0.06719	0.05617	0.09347	0.15469	0.07735
	HPT	0.06716	0.05623	0.09329	0.14829	0.07414
	FPT	0.06663	0.05677	0.09094	0.09037	0.04518
	CPT	0.06663	0.05677	0.09094	—	—
0.2	SPT	0.24691	0.11396	0.28734	0.19369	0.19369
	HPT	0.24686	0.11402	0.28712	0.18561	0.18561
	FPT	0.24608	0.11445	0.28417	0.11296	0.11296
	CPT	0.24608	0.11445	0.28417	—	—
0.3	SPT	0.41229	0.13368	0.36613	0.17884	0.26826
	HPT	0.41223	0.13372	0.36593	0.17137	0.25706
	FPT	0.41129	0.13409	0.36321	0.10427	0.15640
	CPT	0.41129	0.13409	0.36321	—	—
0.5	SPT	0.52203	0.13880	0.36628	0.15501	0.31003
	HPT	0.52197	0.13884	0.36610	0.14854	0.29707
	FPT	0.52093	0.13917	0.36374	0.09037	0.18073
	CPT	0.52093	0.13917	0.36374	—	—
0.9	SPT	0.59086	0.13982	0.34009	0.13364	0.33410
	HPT	0.59079	0.13986	0.33993	0.12805	0.32013
	FPT	0.58970	0.14017	0.33790	0.07790	0.19475
	CPT	0.58970	0.14017	0.33790	—	—

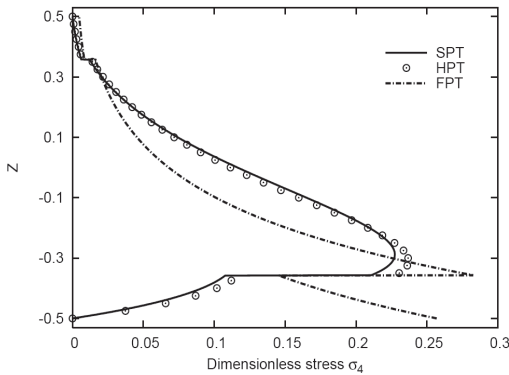


Fig. 2. Distribution of σ_4 through-the-thickness of the (1-5-1) sandwich plate

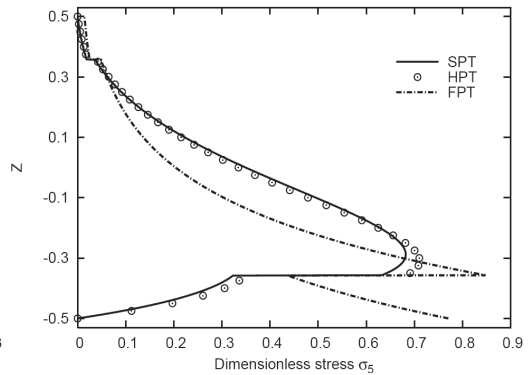


Fig. 3. Distribution of σ_5 through-the-thickness of the (1-5-1) sandwich plate

Figs. 6-8 plot the deflection w vs the time parameter τ of the inhomogeneous (1-5-1) sandwich plate using various theories for different values of aspect ratio b/a , side-to-thickness ratio a/h , and the constitutive parameter ζ , respectively. It is seen that w increases as b/a and a/h increase and decreases as ζ increases (see also Table 1). In addition, w increases to get its local maximum at $\tau \cong 1$, then it decreases to get its local minimum at $\tau \cong 2.3$. After that the deflection is directly increasing as the time parameter τ increases and become constant at $\tau > 15$.

Fig. 9 illustrates the variation of w vs the time parameter τ for different types of inhomogeneous viscoelastic sandwich plates. It can be seen that, the deflection increases with the decrease of the thickness of the core compared with the thickness of the two faces. The deflection becomes constant in the case of the (1-0-1) fully elastic plate. This is expected since there is no viscoelastic core for this plate. Finally, Figs. 10-12 illustrate the variation of in-plane stresses σ_1 , σ_2 , and σ_6 vs τ for different values of thickness Z of the (1-5-1) sandwich plates. The variations of stresses are appears clearly with the variation of the time parameter τ and becomes constant for $\tau > 12$.

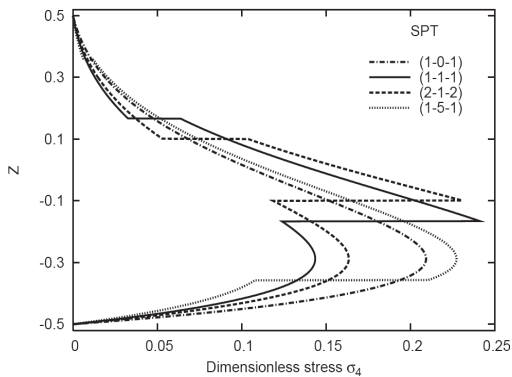


Fig. 4. Distribution of σ_4 through-the-thickness of different sandwich plates

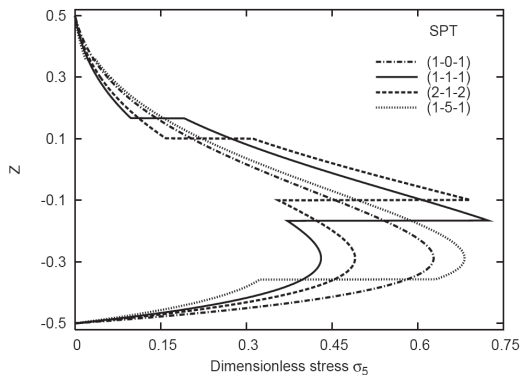


Fig. 5. Distribution of σ_5 through-the-thickness of different sandwich plates

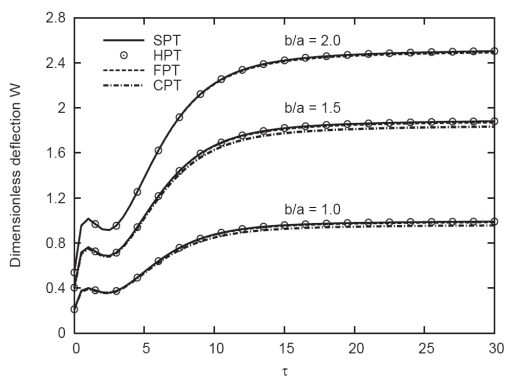


Fig. 6. Variation of w vs τ for different b/a of the (1-5-1) sandwich plate

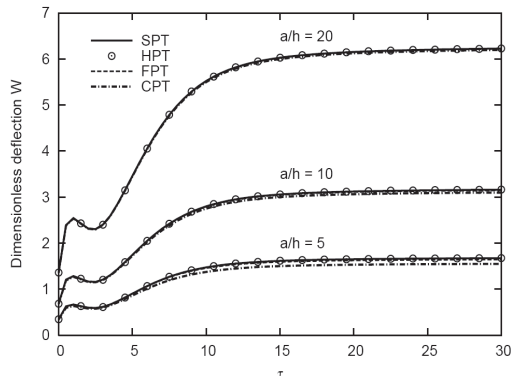


Fig. 7. Variation of w vs τ for different a/h of the (1-5-1) sandwich plate

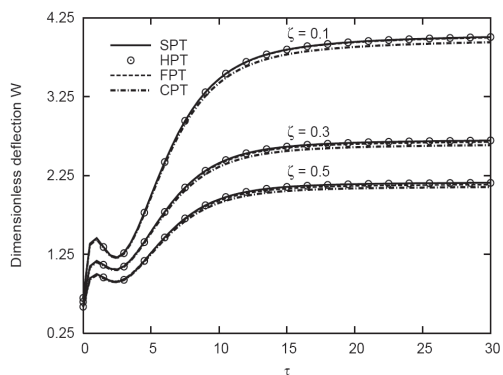


Fig. 8. Variation of w vs τ for different ζ of the (1-5-1) sandwich plate

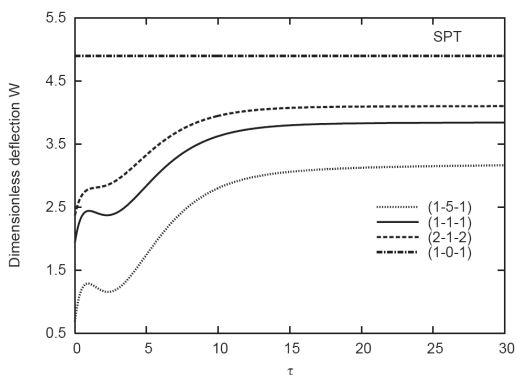


Fig. 9. Variation of w vs τ for different sandwich plates

7. Conclusions

Bending response of inhomogeneous viscoelastic sandwich plates is investigated by using various plate theories. The governing equations of the sinusoidal (SPT), higher-order (HPT), first-order (FPT) and classical (CPT) plate theories are converted into a single system of equations. Analytical solutions are developed using the Navier's procedure and separation of variable technique. The results of SPT are compared with those obtained by HPT, FPT and CPT. The SPT

and HPT contain the same number of dependent variables as in FPT, but results are more accurate prediction of deflections and stresses, and satisfy the zero tangential traction boundary conditions on the surfaces of the plate. However, both SPT and HPT do not require the use of shear correction factors. Numerical computations were carried out to study the effect of time parameter τ on deflections and stresses at different values of aspect ratio b/a , side-to-thickness ratio a/h and constitutive parameter ζ . The obtained results show how the dimensionless stresses and deflection affected with the elastic properties of the layers. It is to be noted that, the variation of dimensionless stresses and deflection versus the time parameter has the high sensitivity at viscoelastic layer. The CPT yields identical stresses with the FPT. In conclusion, the HPT yields results very close to those obtained using SPT which gives accurate results, especially transverse shear stresses, than other theories including HPT.

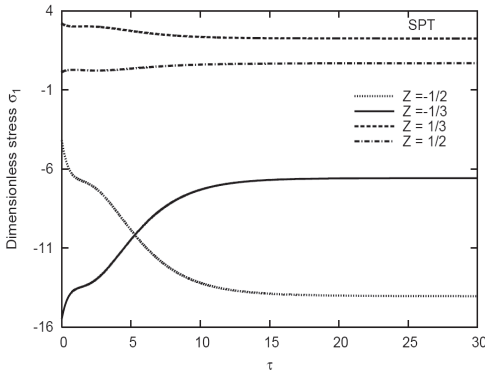


Fig. 10. Variation of σ_1 vs τ for different Z in the (1-5-1) sandwich plate

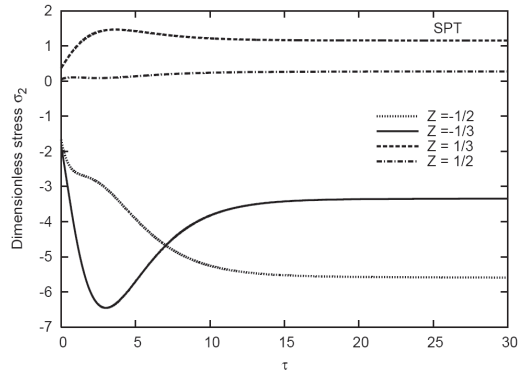


Fig. 11. Variation of σ_2 vs τ for different Z in the (1-5-1) sandwich plate

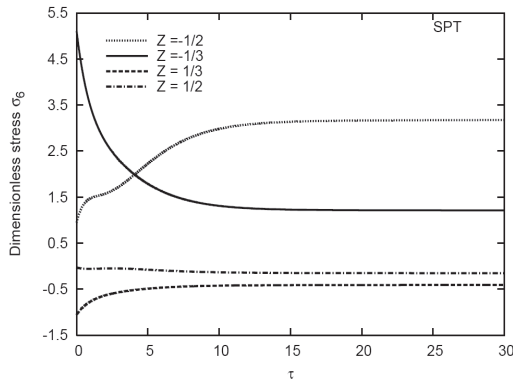


Fig. 12. Variation of σ_6 vs τ for different Z in the (1-5-1) sandwich plate

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