

2219. Free vibration analysis of tapered columns under self-weight using pseudospectral method

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Received 18 April 2016; received in revised form 10 September 2016; accepted 27 October 2016

DOI <https://doi.org/10.21595/jve.2016.17089>

Abstract. This paper deals with the vibration of tapered column which is affected by gravity using a pseudospectral formulation. The formulation is simple and easy-to-implement and is capable of dealing with different end conditions. Numerical examples of the effects of taper, cross section shapes and gravity on the vibration of columns are illustrated. The effectiveness of the pseudospectral method for vibration analysis of tapered heavy columns is validated by comparing the results with numerical techniques such as the numerical initial value method and differential quadrature method.

Keywords: pseudospectral, non-uniform, columns, taper, gravity.

1. Introduction

Elastic columns are a class of important structural components that find wide applications in civil, mechanical and aerospace engineering fields [1]. The strength of an elastic column basically depends on its material and geometrical properties. The material selection and Young's modulus determine whether a column has material non-linearity while the geometric non-linearity arises from non-uniform cross-sectional areas [2]. In conventional column vibration and buckling problems, the self-weight is often neglected and when taken into consideration, the column is referred to as a heavy column [3]. The standing heavy column is fundamental in mechanics and models tall structures and free-standing antennas [4].

Greenhill [5], using Bessel's functions, first investigated the stability of a uniform column due to its own weight. Schafer [6] studied the effect of self-weight on the natural frequencies of a hanging cantilever beam using the Rayleigh-Ritz method. The finite element method was used by Yokoyama [7] to investigate the vibration characteristics of uniform hanging beams under gravity. Virgin et al. [8] performed analytical and experimental studies on the effect of gravity on the vibration of vertical cantilevers. Duan and Wang [3] presented analytical solutions for the buckling of columns including self-weight. Okay et al. [9] applied the variational iteration method to determine the buckling loads and mode shapes of heavy columns under its own weight. An analytic method involving the Fredholm integration method was used by Huan and Li [10] to analyze the buckling behavior of axially nonuniform graded columns. The differential quadrature (DQ) method was used by Mahmoud et al [11] to investigate the effect of column geometry on the natural frequencies and mode shapes. Taha and Essam [12] used the DQ method to study the stability behavior and free vibration of axially loaded tapered columns with elastic end restraints. Recently, Wang [13] used a numerical initial value method to study the influence of gravity as well as taper on the vibration of a standing column.

Although many methods have been presented to analyze problems concerned with taper and self-weight, most of them apply to specific cases determined by the form of the equations. This study presents a simple numerical technique that is capable of handling different cases using the pseudospectral formulation. In terms of effective mathematical techniques, Pseudospectral (PS) methods have been used in recent years for structural engineering analysis. Lee and Schultz [14] applied the Chebyshev PS method to solve the vibration of Timoshenko beams and Mindlin plates.

Yagci et al. [15] used a spectral Chebyshev technique for solving linear and nonlinear beam equations. Sari and Butcher [16] used the PS method for the free vibration analyses of non-rotating and rotating Timoshenko beams with damaged boundaries. In [17], the PS method is used to investigate the dynamic response of Timoshenko beams made of functionally graded materials. A Chebyshev PS method is presented in [18] for the static analysis of the geometrically exact beams undergoing large deflections. However, to the author's knowledge no analytical solution exists for the important problem of the vibration of a standing heavy tapered column. A numerical initial value method proposed in [13] that combines the initial value method [19] with Runge-Kutta method and bisection method is the only available technique present in literature. The important problem demands techniques that are easy to implement and computationally inexpensive.

The intention of this work is to explore the application of a novel formulation of the Chebyshev Pseudospectral method to the vibration analysis of heavy and tapered columns. The method is first validated by computing the frequencies of vibration of non-uniform beams where their density and the flexural rigidity vary along the longitudinal axis. The influence of gravity, taper and gravity and taper on the vibrations of columns is then analyzed and the results are compared with those obtained using exact solutions [20], a Differential Quadrature Method [11] and a numerical initial value method [13].

2. Equations

The vibration of long and slender beams/columns is an important problem in applied mechanics and are generally modeled by the Euler-Bernoulli beam theory [21]. Assuming the effects of rotational inertia and transverse shear deformation to be negligible, the equation for small vibrations of a non-uniform Euler Bernoulli column subjected to an axial force F is given by [13]:

$$\frac{\partial^2}{\partial x^2} \left(EI(x) \frac{\partial^2 y}{\partial x^2} \right) + \frac{\partial}{\partial x} \left(F(x) \frac{\partial y}{\partial x} \right) + \rho(x) \frac{\partial^2 y}{\partial t^2} = 0, \tag{1}$$

where (x, y) are the longitudinal and transverse coordinates of the column with the origin at the base, EI is the flexural rigidity, ρ is the mass per length and t is the time. For a standing column of height L , axial force is given by:

$$F = g \int_x^L \rho(x) dx, \tag{2}$$

where g is the acceleration due to gravity. Introducing $y(x, t) = w(x)e^{ikt}$, $EI(x) = EI_0 l(x)$, $\rho(x) = \rho_0 r(x)$, where EI_0 and ρ_0 are the maximum values of flexural rigidity and mass per length occurring at the base $x = 0$ and k is the frequency of vibration. Normalizing all length by L Eq. (1), becomes:

$$\frac{d^2}{dx^2} \left[l(x) \frac{d^2 w}{dx^2} \right] + \beta \frac{d}{dx} \left[\int_x^1 r(x) \frac{dw}{dx} \right] = \omega^2 r(x) w, \tag{3}$$

where $\omega = kL^2 \sqrt{\rho_0/EI_0}$ and $\beta = g\rho_0 L^3/EI_0$.

Eq. (3) does not have a closed form solution even for a uniform beam/column [21]. Assuming that the column has linear taper with the rigidity and density varying as $l(x) = (1 - cx)^m$, $r(x) = (1 - cx)^n$ where m, n are positive constants and $0 \leq c \leq 1$ representing the degree of taper, Eq. (3) becomes:

$$\frac{d^4w}{dx^4} + b_1(x)\frac{d^3w}{dx^3} + b_2(x)\frac{d^2w}{dx^2} + b_3(x)\frac{dw}{dx} + b_4(x)w = 0, \quad (4)$$

where:

$$b_1(x) = -\frac{2mc}{1-cx}, \quad (c \neq 0),$$

$$b_2(x) = \frac{m(m-1)c^2}{(1-cx)^2} + \frac{\beta\{(1-cx)^{n+1} - (1-c)^{n+1}\}}{c(n+1)(1-cx)^m} = \beta(1-x), \quad (c = 0),$$

$$b_3(x) = -\beta(1-cx)^{n-m},$$

$$b_4(x) = -\omega^2(1-cx)^{n-m}.$$

For clamped-free (C-F) columns with clamped end $x = 0$ and free end $x = 1$, the boundary conditions can be written as:

$$w = \frac{dw}{dx} = 0, \quad x = 0, \quad (5)$$

$$\frac{d^2w}{dx^2} = 0, \quad \frac{d}{dx} \left[l(x) \frac{d^2w}{dx^2} \right] = 0, \quad x = 1. \quad (6)$$

If $c = 0$, the column is uniform and for $c = 1$, the column has a pointy tip.

For $c \neq 1$, Eq. (6) reduces to:

$$\frac{d^2w}{dx^2} = 0, \quad \frac{d^3w}{dx^3} = 0, \quad x = 1. \quad (7)$$

Eq. (4) is solved subject to the boundary conditions given by Eq. (5) and Eq. (6) for different values of m, n . Though the equations can be solved for general values of m, n , we consider some special cases which correspond to those shown in Fig. 1. In the case of a solid tapered column of circular cross section, $m = 4, n = 2$, while for a solid column of constant thickness and tapered sides $m = 1, n = 1$ or $m = 3, n = 1$. If the column vibrates about the axis A-A which is perpendicular to the thickness direction $m = 1, n = 1$ and if the column vibrates about the axis B-B which is parallel to the thickness direction $m = 3, n = 1$ [13].

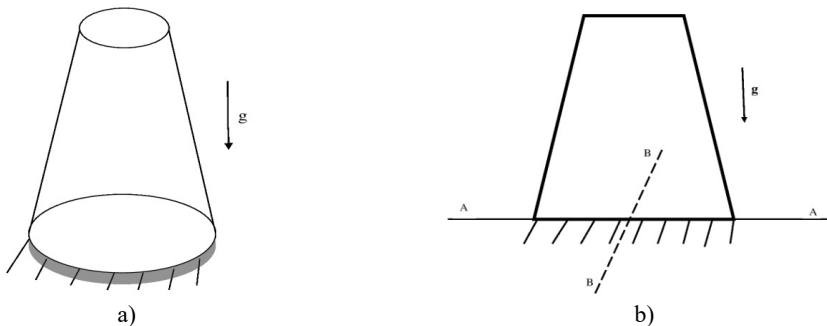


Fig. 1. a) Tapered column of circular cross section, b) solid column of constant thickness and tapered sides

2.1. Solution procedure

In Chebyshev PS method, the Chebyshev polynomials are employed as the trial functions for the discretization of the unknown function namely w and the Gauss-Chebyshev-Lobatto points are employed as the collocation points at which the residuals are minimized. The physical domain

$0 \leq x \leq 1$ is transformed into $-1 \leq X \leq 1$ by the transformation $X = 2x - 1$. With this transformation Eq. (4) reduces to:

$$\frac{d^4 w}{dX^4} + B_1(X) \frac{d^3 w}{dX^3} + B_2(X) \frac{d^2 w}{dX^2} + B_3(X) \frac{dw}{dX} + B_4(X)w = 0, \tag{8}$$

where $B_i(X) = b_i(x)/2^i$, $i = 1, 2, 3, 4$.

We assume:

$$w = \sum_{k=0}^N a_k T_k(X), \tag{9}$$

where a_k , ($k = 0, \dots, N$) are unknown constants and $T_k(X)$ ($k = 0, 1, \dots, N$) are Chebyshev polynomials defined by [22] $T_k(X) = \cos(k \cos^{-1}(X)) = \cos(k\theta)$ for $k = 0, 1, 2, \dots$, where $\theta = \cos^{-1}(X)$.

The transformation $X = \cos\theta$ converts the Chebyshev series into a Fourier cosine series. In the proposed methodology, we compute the basis functions and their derivatives using trigonometric functions:

$$i.e. X = \cos\theta \leftrightarrow \theta = \cos^{-1}(X), \quad X \in [-1 \ 1], \quad t \in [0 \ \pi]. \tag{10}$$

The elementary identity:

$$\frac{d}{dX} \leftrightarrow \left(-\frac{1}{\sin\theta}\right) \frac{d}{d\theta} \tag{11}$$

is repeatedly applied to convert Eq. (8) into an equivalent differential equation on $\theta \in [0 \ \pi]$. As $\theta \rightarrow 0, \pi$ the derivatives are evaluated using the Taylor expansions of both numerator and denominator about their common zero. Substituting the value of w given by Eq. (9) into Eq. (8) and using Eq. (10), Eq. (11) the equivalent differential equation is collocated at:

$$\theta_i = \frac{(N-i)\pi}{N}, \quad i = 1, \dots, N-3, \quad (0, \pi), \tag{12}$$

yielding a system of $(N-3)$ equations in $N+1$ unknowns a_k . Imposing the boundary conditions given by Eq. (5), Eq. (6) we get a system of 4 equations in $(N+1)$ unknowns. The resulting $(N+1)$ by $(N+1)$ system of equations is expressed as a matrix eigenvalue problem and solved using a standard eigensolver.

3. Numerical results and discussion

In this section, we study the convergence behavior of the PS method first and then consider some numerical examples to validate the efficiency of the Pseudospectral method. The first numerical example is devoted to a beam with constant thickness and linearly tapered width with clamped base and no tip mass. The second is concerned with the vibration of a hanging uniform column under self-weight. The third example considers the free vibration of non-uniform column with no self-weight. The last example concerns the influence of gravity and taper on the vibration of a standing column.

3.1. Convergence behavior of PS method

As a case study, the convergence behavior of the non-dimensional frequency parameter (ω)

for the first five modes of a non-uniform beam whose cross section is an open web or tower type [20] is considered. To highlight this, we consider the clamped-free boundary condition with a taper ratio of $c = 0.1$ in the absence of gravity. Taking the exact values of (ω) given in [20] as base values, we compute the values of (ω) for N varying from 20 to 28. The results obtained to six-digit precision are presented in Table 1. We observe from the table that the converged results have good accuracy in comparison with those of [20].

Table 1. Non-dimensional vibration frequencies for $m = 4, n = 0$

(ω)	N					
	20	22	24	26	27	28
ω_1	3.376722	3.376722	3.376722	3.376722	3.376722	3.376722
ω_2	20.248149	20.248149	20.248149	20.248149	20.248149	20.248149
ω_3	55.966597	55.966595	55.966595	55.966595	55.966595	55.966595
ω_4	109.302542	109.301970	109.301978	109.301978	109.301978	109.301978
ω_5	180.403245	180.432576	180.430075	180.430210	180.430205	180.430205

3.2. Vibration of a class of Non-uniform beams in the absence of gravity

In [20], new exact solutions were presented for a class of non-uniform beams whose density and flexural rigidity vary along the longitudinal axis. To assess the numerical accuracy of the proposed PS method, we obtain the numerical solutions for a class of non-uniform beams presented in [20]. Ignoring rotational inertia and shear deformation, the Euler-Bernoulli small deflection beam equation is obtained by taking $F \equiv 0$ in Eq. (1). To test the validity of the method for different end conditions we consider Clamped-Free (C-F), Pinned-Pinned (P-P) and Free-Free (F-F) end conditions. At clamped and free ends the boundary conditions can be used from Eqs. (5-6). For a pinned end the conditions are $w = 0, d^2w/dx^2 = 0$.

The computations in the PS method (PSM) are carried out using $N = 30$ and the results obtained are presented in Table 2 along with the exact values [20]. The exact values given in [20] are obtained through power function solutions. The results are presented in the form of tables to highlight the numerical accuracy and for easy comparison.

Table 2. Non-dimensional vibration frequencies for $m = 4, n = 0$

Taper ratio c	0.1		0.3		0.7	
Boundary conditions	Wang [20]	PSM	Wang [20]	PSM	Wang [20]	PSM
C-F	3.3767	3.376722	3.0751	3.075097	2.3151	2.315075
	20.248	20.248149	16.680	16.680409	9.3906	9.390641
	55.367	55.966595	44.733	44.733218	22.898	22.897800
	109.30	109.301978	86.631	86.630457	42.975	42.974975
	180.43	180.430205	142.50	142.494970	69.698	69.698967
F-F	20.162	20.161464	15.890	15.889909	7.9004	7.900440
	55.566	55.566239	43.715	43.714562	21.303	21.302894
	108.92	108.923974	86.625	85.625417	41.369	41.369409
	180.05	180.050691	141.49	141.487640	68.084	68.085117
	268.96	268.959692	211.31	211.313335	101.46	101.465340
P-P	8.8895	8.889481	6.9698	6.969833	3.2686	3.268554
	35.570	35.569613	27.985	27.984517	13.647	13.646465
	80.026	80.026031	62.915	62.914773	30.430	30.430199
	142.26	142.263141	111.80	111.800265	53.836	53.836215
	222.28	222.281555	174.65	174.646486	83.894	83.894116

3.3. Vibration of a uniform column under self-weight

The equation for the vibration of a uniform column under self-weight is obtained by taking $c = 0$ in Eq. (4). The present scheme is applied to solve the resulting equation using $N = 25$. The

results obtained are compared with the values given in [4]. It is to be noted that negative values of β denotes a hanging column. The first two square of non-dimensional frequency values (ω^2) obtained using the present method with the base fixed and the top experiencing zero moment and shear is given in Table 3. The closeness of the results obtained using PSM with that of the results obtained using the method of [19] is seen in the table.

Table 3. Square of Non-dimensional frequency values of a column under self-weight

ω^2	β							
	0		-20		-50		-100	
	Wang [4]	PSM	Wang [4]	PSM	Wang [4]	PSM	Wang [4]	PSM
ω_1^2	12.362	12.362363	43.53	43.530595	89.65	89.653010	165.6	165.604518
ω_2^2	485.50	485.518818	657.7	657.648489	913.1	913.078445	1333	1332.919282

3.4. Free Vibration of non-uniform column

The governing equation of a non-uniform column varying with bending stiffness given by $1 + \alpha x$ as in [11] under no self-weight is obtained by taking $n = 0, m = 1, \beta = 0$ and $c = -\alpha$ in Eqn. (4). In [11], the free vibration of non-uniform column was considered efficiently using the Differential Quadrature Method(DQM). Two variants of DQM namely the modifying weighting coefficient matrices (MWCM) method and substituting boundary conditions into governing equations (SBCGE) techniques were used to treat different types of boundary conditions in [11]. The simple-simple (S-S) supports/pinned-pinned (P-P) supports and clamped-simple (C-S) supports at bottom and top were treated using MWCM technique while the clamped-clamped (C-C) supports and clamped-free (C-F) supports were treated using SBCGE technique. However, the present Pseudospectral formulation is capable of treating the different boundary conditions with ease to compute the non-dimensional frequencies (ω) of non-uniform columns. The computed first three frequency values for $\alpha = -0.5, 0.5$ in the case of S-S and C-C supports are presented in Table 4 along with the corresponding values of [11]. It is to be noted that the computations were carried out using $N = 15$ as in [11] for a fair comparison. The results show that the present formulation of PS method is an efficient method in solving the free vibration of non-uniform columns with good accuracy.

Table 4. Non-dimensional frequency values of non-uniform column

Non-dimensional frequency values	$\alpha = -0.5$		$\alpha = 0.5$		$\alpha = -0.5$		$\alpha = 0.5$	
	MWCM [11]	PSM	MWCM [11]	PSM	SBCGE [11]	PSM	SBCGE [11]	PSM
	(S-S)		(S-S)		(C-C)		(C-C)	
ω_1	8.479	8.479450	11.003	11.003523	19.098	19.098602	24.888	24.888283
ω_2	33.834	33.834311	43.976	43.976308	52.709	52.709121	68.633	68.633917
ω_3	76.065	76.065006	98.919	98.919180	103.38	103.383220	134.57	134.575354

3.5. Vibration of a standing tapered heavy column

In [13], the stability and natural vibration of a standing tapered vertical column under its own weight is studied. The method consists in using a simple initial value method combined with interpolation using the Runge-Kutta method to obtain the frequencies of vibration. The present Pseudospectral method is much simpler to implement for computer usage. Though the method is suitable for general values of cross-section shape parameters (m, n), we consider the values (3, 1), (1, 1) and (4, 2) for a fair comparison. The non-dimensional vibration frequencies are computed for the taper values of $c = 0.1, 0.3$ and 0.7 with the gravity parameter (β) taking the values 0, 2.5 and 7.5. The computations in the PSM are carried out using $N = 25$. The results obtained are presented in Tables 5-7 and are compared with the corresponding values given in [13]. In [13], there is a misprint in the 2nd frequency values corresponding to $c = 0.7, \beta = 2.5$ and $c = 0.3,$

$\beta = 7.5$ for $m = 3, n = 1$ and the correct values obtained using PSM are given in Table 5. It is observed that as the gravity effect (β) increases, the frequencies decrease until the fundamental frequency is almost zero, at which stage the column buckles. In addition, the closeness of the values with those of [13] brings out the simplicity and accuracy of the present method.

Table 5. Non-dimensional frequencies for $m = 3, n = 1$

β	c					
	0		0.3		0.7	
	Wang [13]	PSM	Wang [13]	PSM	Wang [13]	PSM
0	3.5587	3.558702	3.667	3.666749	4.0817	4.081714
	21.338	21.338102	19.889	19.880606	16.625	16.625269
	58.980	58.979904	53.322	53.322198	40.588	40.587991
2.5	2.9485	2.948465	3.0621	3.062087	3.4971	3.497114
	20.837	20.836806	19.369	19.368556	14.085	16.085256
	58.470	58.469500	52.806	52.805953	40.051	40.051452
7.5	0.8466	0.846644	1.0938	1.093815	1.8155	1.815472
	19.794	19.794401	19.300	18.300150	14.947	14.946583
	57.433	57.432830	51.756	51.755963	38.955	38.954952

Table 6. Non-dimensional frequencies for $m = 1, n = 1$

β	c					
	0.1		0.3		0.7	
	Wang [13]	PSM	Wang [13]	PSM	Wang [13]	PSM
0	3.6310	3.631027	3.9160	3.916033	4.9317	4.931642
	22.254	22.254029	22.786	22.785958	24.687	24.687279
	61.910	61.909628	62.463	62.436120	64.527	64.526628
2.5	3.0376	3.037632	3.3638	3.363823	4.4793	4.479281
	21.771	21.770841	22.333	22.332727	24.315	24.315225
	61.416	61.416302	61.976	61.975560	64.153	64.152600
7.5	1.1326	1.132558	1.8024	1.802415	3.3960	3.395995
	20.769	20.769043	21.396	21.396124	23.553	23.552729
	60.415	60.415453	61.042	61.042332	63.397	63.397027

Table 7. Non-dimensional frequencies for $m = 4, n = 2$

β	c					
	0.1		0.3		0.7	
	Wang [13]	PSM	Wang [13]	PSM	Wang [13]	PSM
0	3.6737	3.673701	4.0669	4.066932	5.5093	5.509268
	21.550	21.550253	20.556	20.555506	18.641	18.641218
	58.189	59.188637	54.015	54.015186	42.810	42.810666
2.5	3.0821	3.082123	3.5181	3.518063	5.0536	5.053637
	21.062	21.062033	20.085	20.084902	18.212	18.212036
	58.693	58.692572	53.543	53.543037	42.385	42.385379
7.5	1.2137	1.213739	2.0036	2.003562	3.9836	3.983635
	20.048	20.048376	19.108	19.107981	17.322	17.321630
	57.686	57.685602	57.584	57.584453	41.521	41.521269

4. Conclusions

Typically, the free vibration frequencies of a non-uniform gravity loaded Euler-Bernoulli column/beam is obtained numerically as the governing fourth order differential equation with variable coefficients does not yield any closed form solutions. The numerical technique-Chebyshev Pseudospectral method explored in this paper introduces a novel formulation of the method in which the basis functions and their derivatives are computed using

trigonometric functions. The stability of the method is first studied by obtaining the vibration frequencies of a linearly tapered beam in the absence of gravity. The accuracy of the method is further tested against the exact solutions of a class of non-uniform beams in the absence of gravity, solutions obtained using two variants of DQM for a non-uniform column under different end conditions and also against the solutions obtained using an initial value method in the case of a uniform column under self-weight. Finally, the proposed method is used to find the vibration frequencies of a standing linearly tapered heavy column. A comparison of the results obtained with those of the numerical initial value method shows that the proposed technique is an efficient and reliable method in handling vibration columns of elastic columns/beams. It is also possible to extend the technique to other tapers and consider inclusion of shear effects as in a Timoshenko column.

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