39. Benchmark solutions of the free vibration of simply supported laminated composite plates

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Abstract. Due to the high specific strength and stiffness, laminated composite plates, especially mid-plane symmetric laminated composite plates, are frequently used as structural component in aeronautical and aerospace engineering. Since obtaining analytic solutions are difficult even for simply supported mid-plane symmetric laminated composite plates, numerical methods have to be used to obtain approximate solutions. To evaluate various numerical methods, benchmark solutions are needed. In this article, highly accurate frequencies of simply supported angle-ply mid-plane symmetric laminated composite plates with two sets of equivalent material properties are obtained by the modified differential quadrature method and presented to serve as the benchmark solutions.

Keywords: laminated composite plate, equivalent material properties; benchmark solution, free vibration; modified differential quadrature method.

1. Introduction

Due to the high specific strength and stiffness, laminated composite plates are frequently used as structural component in aeronautical and aerospace engineering. Their static, buckling and free vibration behavior is of important to the designers and thus has been received great attentions [1, 2]. Among various types of laminations, the mid-plane symmetric laminates are widely used in practice. The de-coupling of in-plane and out-of-plane deformation makes the production of a flat plate as well as analysis much simpler than the general laminations.

In free vibration analysis, the laminated composite plates are usually equivalent to anisotropic plates. Analytical solutions are rarely available even for rectangular anisotropic plates with simple supported boundary conditions. Therefore, various approximate approaches [1-4] and numerical methods [5-9] have been employed for solutions.

In literature, two equivalent ways of expressing the material properties are commonly used. Take the E-glass/epoxy (E/E) material as an example, the material properties expressed in one way, called the material system I (MS-I), are $E_1 = 60.7$ GPa, $E_2 = 24.8$ GPa, $G_{12} = 12.0$ GPa and $\nu_{12} = 0.23$ [3, 8, 10], and the material properties expressed in the other way, called the material system II (MS-II), are $E_1/E_2 = 2.45$, $G_{12}/E_2 = 0.48$ and $\nu_{12} = 0.23$ [4, 6, 7]. Researchers often do not distinguish one set of material properties from the other since they are regarded equivalent. The choice mainly depends on their personal preference. In references [4, 6, 7], the results are obtained based on the MS-II, but compared with the upper bound solutions with the MS-I [3]. In reference [8], the material properties of MS-I are given, but the data are actually obtained with the ones with MS-II. Occasionally this might cause mis-understanding to the readers, although it is not difficult to tell that MS-II is actually used in their calculation by looking at the exact solutions of special orthotropic rectangular plates, since the exact solutions for the two sets of equivalent material systems are slightly different and the corresponding Ritz solutions of special orthotropic plates reported in [3] are also exact solutions [10]. The difference in solutions for the laminated composite plates with the two sets of equivalent material properties is really small and negligible from the practical point of view.

From the computational point of view, however, the small difference may be important in testing the accuracy and efficiency of new numerical methods. Very accurate benchmark solutions are required in such cases. The data reported in [3, 4] are not very accurate due to either the lower rate of convergence of the method or the extra constraints implicitly enforced in the test functions [11]. More terms in the series of the test functions are needed to obtained solutions with higher accuracy by using the Ritz method.

The primary objective of this paper is to provide highly accurate benchmark frequencies for simply supported square laminated composite plates with two sets of equivalent material properties. The modified differential quadrature method proposed by the author is used to obtain accurate solutions. The slight difference in the frequencies of the mid-plane symmetric laminates with two sets of equivalent material systems is clearly demonstrated.

2. Basic equations and solution procedures

2.1. Governing equation and expression of boundary condition

Denote the length, width and total thickness of the rectangular laminated composite plate by a, b, and h. The governing equation for the free vibration analysis of a mid-plane symmetric laminated composite plate is given by:

$$\bar{D}_{11}\frac{\partial^4 w}{\partial x^4}+4\bar{D}_{16}\frac{\partial^4 w}{\partial x^3\partial y}+2(\bar{D}_{12}+2\bar{D}_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+4\bar{D}_{26}\frac{\partial^4 w}{\partial x\partial y^3}+\bar{D}_{22}\frac{\partial^4 w}{\partial y^4}=\rho h\omega^2 w, \quad (1)$$

where $\bar{D}_{ij}$ are the effective bending and twisting stiffness [12], $w(x,y)$ is the deflection, ρ and ω are the mass density and circular frequency, respectively.

The expressions of simply supported boundary conditions are:

$$\begin{cases} w=0, \quad M_x=0(x=0,a), \\ w=0, \quad M_y=0(y=0,b), \end{cases} \quad (2)$$

where the expressions of bending moments M_x and M_y are:

$$\begin{cases} M_x=\bar{D}_{11}\frac{\partial^2 w}{\partial x^2}+2\bar{D}_{16}\frac{\partial^2 w}{\partial x\partial y}+\bar{D}_{12}\frac{\partial^2 w}{\partial y^2}, \\ M_y=\bar{D}_{12}\frac{\partial^2 w}{\partial x^2}+2\bar{D}_{26}\frac{\partial^2 w}{\partial x\partial y}+\bar{D}_{22}\frac{\partial^2 w}{\partial y^2}. \end{cases} \quad (3)$$

2.2. Modified differential quadrature method and solution procedures

For completeness considerations, the modified differential quadrature method (modified DQM) and solution procedures are briefly introduced.

Denote N_x and N_y the numbers of grid points in x and y directions, and (x_i, y_j) ($i = 1, 2,..., N_x$; $j = 1, 2,..., N_y$) the grid points. In the modified DQM, two additional derivative degrees of freedom at end points are introduced by using the method of modification of weighting coefficient-3 (MMWC-3) proposed by the author [9].

For simplicity and demonstration of the method, take a one-dimensional problem as an example. In the ordinary differential quadrature method, the first order derivative of the solution $w(x)$ with respect to x at grid point x_i is approximated as:

$$\left.\frac{dw}{dx}\right|_{x=x_i}=\sum_{j=1}^{N_x} A_{ij}^x w_j, \quad (i=1,2,\dots,N_x), \quad (4)$$

where A_{ij}^x is called the weighting coefficient, which can be explicitly computed by:

$$A_{ij}^x = \begin{cases} \prod_{k=1,k\neq i,j}^{N_x} \dfrac{x_i - x_k}{\prod_{k=1,k\neq j}^{N_x}(x_j - x_k)}, & (i \neq j), \\ \sum_{k=1,k\neq i}^{N_x} \dfrac{1}{(x_i - x_k)}, & (i = j), \end{cases} \quad (i,j = 1,2,\dots,N_x). \tag{5}$$

To apply multiple boundary conditions rigorously, two additional degrees of freedom (DOFs) are introduced during formulation the weighting coefficient of the second order derivatives at two end points by using the MMWC-3 [9], namely:

$$\begin{aligned} \left.\frac{d^2w}{dx^2}\right|_{x=x_i} &= \sum_{k=1}^{N_x} A_{ik}^x \left.\frac{dw}{dx}\right|_{x=x_k} = \sum_{k=2}^{N_x-1} A_{ik}^x \sum_{j=1}^{N_x} A_{kj}^x w_j + A_{i1}^x w_1' + A_{iN}^x w_N' \\ &= \sum_{j=1}^{N_x} \tilde{B}_{ij}^x w_j + A_{i1}^x w_1' + A_{iN}^x w_N' = \sum_{j=1}^{N+2} \tilde{B}_{ij}^x \delta_j, \quad (i = 1, N_x), \end{aligned} \tag{6}$$

where $\delta_j = w_j$ ($j = 1, 2, ..., N_x$), $\delta_{N+1} = w_1'$, $\delta_{N+2} = w'_N$, $\tilde{B}_{i(N+1)}^x = A_{i1}^x$, $\tilde{B}_{i(N+2)}^x = A_{iN}^x$.

At all inner points, the weighting coefficients of the second order derivative are the same as the ordinary DQM, namely:

$$\begin{cases} \tilde{B}_{ij}^x = B_{ij}^x = \sum_{k=1}^{N_x} A_{ik}^x A_{kj}^x, & (i = 2,3,\dots,N_x - 1, \quad j = 1,2,\dots,N_x), \\ \tilde{B}_{ij}^x = 0, & (i = 2,3,\dots,N_x - 1, \quad j = N_x + 1, N_x + 2), \\ B_{ij}^x = \sum_{k=1}^{N_x} A_{ik}^x A_{kj}^x, \quad \tilde{B}_{ij}^x = \sum_{k=2}^{N_x-1} A_{ik}^x A_{kj}^x, & (i = 1, N_x, \quad j = 1,2,\dots,N_x). \end{cases} \tag{7}$$

In the modified DQM, the weighting coefficients of the third and the fourth order derivatives, denoted by $\tilde{C}_{ij}^x$, $\tilde{D}_{ij}^x$ are computed by:

$$\tilde{C}_{ij}^x = \sum_{k=1}^{N_x} A_{ik}^x \tilde{B}_{kj}^x, \quad (i = 1,2,\dots,N_x, \ \ j = 1,2,\dots,N_x + 2), \tag{8}$$

$$\tilde{D}_{ij}^x = \sum_{k=1}^{N_x} B_{ik}^x \tilde{B}_{kj}^x, \quad (i = 1,2,\dots,N_x, \ \ j = 1,2,\dots,N_x + 2). \tag{9}$$

The weighting coefficients of the first to fourth-order derivatives with respect to y can be calculated in a similar way, simply replacing x and N_x in Eq. (5) to Eq. (9) by y and N_y. Since only square plates ($a = b$) are considered, thus $N_x = N_y = N$. In terms of the modified differential quadrature (DQ), the bending moments at corresponding boundary points can be expressed as:

$$(M_x)_{il} = \bar{D}_{11}\sum_{k=1}^{N+2}\tilde{B}_{ik}^{x}\,\tilde{w}_{kl} + 2\bar{D}_{16}\sum_{j=1}^{N}\sum_{k=1}^{N}A_{ij}^{x}A_{lk}^{y}\tilde{w}_{jk} + \bar{D}_{12}\sum_{k=1}^{N+2}\tilde{B}_{lk}^{y}\tilde{w}_{ik}, \quad (i = 1, N, \quad l = 1,2,\ldots,N), \tag{10}$$

$$(M_y)_{il} = \bar{D}_{12}\sum_{k=1}^{N+2}\tilde{B}_{ik}^{x}\,\tilde{w}_{kl} + 2\bar{D}_{26}\sum_{j=1}^{N}\sum_{k=1}^{N}A_{ij}^{x}A_{lk}^{y}\tilde{w}_{jk} + \bar{D}_{22}\sum_{k=1}^{N+2}\tilde{B}_{lk}^{y}\tilde{w}_{ik}, \quad (i = 1,2,\ldots,N, \quad l = 1, N). \tag{11}$$

In terms of the DQ, the governing equation at all grid points can be expressed as:

$$\begin{aligned}&\bar{D}_{11}\sum_{k=1}^{N+2}\tilde{D}_{ik}^{x}\bar{w}_{kl} + 4\bar{D}_{16}\sum_{j=1}^{N+2}\sum_{k=1}^{N}\tilde{C}_{ij}^{x}A_{lk}^{y}\bar{w}_{jk} + 2(\bar{D}_{12} + 2\bar{D}_{66})\sum_{j=1}^{N}\sum_{k=1}^{N}B_{ij}^{x}B_{lk}^{y}\bar{w}_{jk} \\ &+4\bar{D}_{26}\sum_{j=1}^{N}\sum_{k=1}^{N+2}A_{ij}^{x}\tilde{C}_{lk}^{y}\bar{w}_{jk} + \bar{D}_{22}\sum_{k=1}^{N+2}\tilde{D}_{lk}^{y}\bar{w}_{ik} = \rho h\omega^2 w_{il}, \quad (i, l = 1,2,\ldots,N),\end{aligned} \tag{12}$$

where superscripts x and y mean that the weighting coefficients of the corresponding derivatives are taken with respect to x and y, $\bar{w}_{ik}$ contains the deflection w_{il} as well as the first-order derivative with respect to x or y along boundary points, introduced by the method of modification of weighting coefficient-3 (MMWC-3), $\tilde{w}_{kl}$, $\tilde{w}_{jk}$ and $\tilde{w}_{ik}$ are only a part of $\bar{w}_{ik}$. There are $(N + 2) \times (N + 2) - 4$ degrees of freedom (DOFs) in total. From Eq. (7), it is clearly seen that B_{ij}^{x} $(i = 1, N)$ are different from $\tilde{B}_{ij}^{x}$ $(i = 1, N)$, and B_{lk}^{y} $(l = 1, N)$ are different from $\tilde{B}_{lk}^{y}$ $(l = 1, N)$.

The bending moment equation is placed at the position where the DOF of the first-order derivative with respect to x or y at corresponding boundary point is. Enforcing the simply supported boundary conditions rigorously yields following partitioned matrix equations, namely:

$$\begin{bmatrix}[K_{\alpha\alpha}] & [K_{\alpha\beta}] \\ [K_{\beta\alpha}] & [K_{\beta\beta}]\end{bmatrix}\begin{Bmatrix}\{w_\alpha\} \\ \{w_\beta\}\end{Bmatrix} = \Omega^2\begin{bmatrix}[I] & [0] \\ [0] & [0]\end{bmatrix}\begin{Bmatrix}\{w_\alpha\} \\ \{w_\beta\}\end{Bmatrix}, \tag{13}$$

where $\Omega = \omega a^2\sqrt{\rho h/D_0}$ is called the frequency parameter, $D_0 = E_1 h^3/[12(1 - \nu_{12}\nu_{21})]$, E_1, ν_{12} and ν_{21} are the modulus of elasticity in the fiber direction, as well as the major and minor Poisson's ratios, respectively. The vector $\{w_\alpha\}$ contains only the non-zero DOFs of the deflection at all inner grid points and its dimension is $(N - 2) \times (N - 2)$.

After eliminating $\{w_\beta\}$, Eq. (13) can be rewritten in the following matrix equation:

$$[\bar{K}]\{w_\alpha\} = \Omega^2[I]\{w_\alpha\}, \tag{14}$$

where $[\bar{K}] = [K_{\alpha\alpha} - K_{\alpha\beta}K_{\beta\beta}^{-1}K_{\beta\alpha}]$.

Solving Eq. (14) by a standard eigen-solver yields the frequency parameters.

To achieve the fastest rate of convergence and obtain reliable and accurate solutions, following grid points are used in the modified DQM:

$$x_k = y_k = \frac{a\left[1 - \cos\left(\dfrac{(k-1)\pi}{N-1}\right)\right]}{2}, \quad (k = 1,2, \ldots, N, \quad a = b). \tag{15}$$

The exact frequency parameters (Ω) for especially orthotropic rectangular plates can be calculated analytically by [1, 3]:

$$\Omega_{exact} = \omega a^2 \sqrt{\frac{\rho h}{D_0}} = \pi^2 \sqrt{\frac{\bar{D}_{11}}{D_0} m^4 + 2\left(\frac{\bar{D}_{12}}{D_0} + 2\frac{\bar{D}_{66}}{D_0}\right)\left(\frac{a}{b}\right)^2 m^2 n^2 + \frac{\bar{D}_{22}}{D_0}\left(\frac{a}{b}\right)^2 n^4}, \quad (m, n = 1,2, \dots), \tag{16}$$

where m and n are the half wave number of the vibration mode in x and y directions, respectively.

3. Results and discussion

Three materials of lamina, i.e., E-glass/epoxy (E/E), Boron/epoxy (B/E) and Graphite/epoxy (G/E), are considered. The material parameters directly taken from [3, 4] are listed in Table 1. For each material, two sets of equivalent material constants are given. Among the three materials, Graphite/epoxy exhibits the highest anisotropy, since E_1/E_2 is the largest.

Table 1. Material property of two sets of equivalent material constants

Materials	Material system I (MS-I) [3]				Material system II (MS-II) [4]		
	E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	ν_{12}	E_1/E_2	G_{12}/E_2	ν_{12}
E/E	60.7	24.8	12.0	0.23	2.45	0.48	0.23
B/E	209.	19.0	6.40	0.21	11.0	0.34	0.21
G/E	138.	8.96	7.10	0.30	15.4	0.79	0.30

Denote θ the fiber orientation angle. Four angles, i.e., $\theta = 0°$, 15°, 30° and 45°, are considered. The relative bending-twisting coupling coefficients D_{16}/D_0 and D_{26}/D_0, which reflect the degrees of anisotropy, are listed in Table 2.

Table 2. Relative bending-twisting coefficients of angle-ply ($\theta/-\theta/\theta$) laminated plates

$\theta°$	E-glass/epoxy (E/E)		Boron/epoxy (B/E)		Graphite/epoxy (G/E)	
	D_{16}/D_0	D_{26}/D_0	D_{16}/D_0	D_{26}/D_0	D_{16}/D_0	D_{26}/D_0
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
15	0.122312	0.012555	0.214391	0.012882	0.205801	0.027967
30	0.176432	0.079666	0.297578	0.096069	0.291366	0.113533
45	0.147858	0.147858	0.227273	0.227273	0.233768	0.233768

It is seen that D_{16}/D_0 is the largest when $\theta = 30°$. This perhaps is the reason why the convergence study is performed for $\theta = 30°$ in [3], since the higher the anisotropy, the lower the rate of convergence for various approximate and numerical methods. Although D_{16}/D_0 is the second largest when $\theta = 45°$, however, D_{26}/D_0 is the largest. Thus, convergence studies are performed for both $\theta = 30°$and $\theta = 45°$ in present investigations. Corresponding results are listed in Table 3 and Table 4, respectively. Mid-plane symmetric angle-ply square plates with all edges simply supported, denoted by SSSS, are investigated.

From Table 3 and Table 4, it is clearly seen that the rate of convergence of the DQM is high. The rate of convergence of the DQM for $\theta = 30°$ is higher than the one for $\theta = 45°$. This indicates that the anisotropy of the (45°/–45°/45°) square plates is higher than the one of the (30°/–30°/30°) square plates for the same material and the anisotropy of the graphite/epoxy square plates with (45°/–45°/45°) is the highest.

To ensure the high accuracy of solutions, the frequency parameters of three-layer angle-ply ($\theta/-\theta/\theta$) square plates with all edges simply supported are obtained by the modified DQM with 31×31 grid points and are presented in Tables 5-7. The DQ solutions contain results using two sets of equivalent material constants listed in Table 1 and are all below the upper bound solutions cited from [3]. Note that the Ritz data reported in [3] are exact only for the case of $\theta = 0°$.

In Table 5, Table 6, and Table 7, the exact solutions for $\theta = 0°$ are re-computed by using Eq. (16) with the corresponding material constants, since the existing exact solutions are only accurate to two places of decimals. It is observed that the DQ data are exactly the same as the re-computed exact solutions. The exact solutions with MS-I of materials E/E and G/E are slightly higher than the corresponding ones with MS-II, and the exact solutions with MS-I of material B/E are slightly lower than the corresponding ones with MS-II. This trend remains the same in the DQ solutions for other fiber orientation angles. It seems that this trend is mainly caused by the difference of G_{12}, since G_{12} in MS-I of materials E/E and G/E is also slightly larger than G_{12} in MS-II and G_{12} in MS-I of material B/E is smaller than G_{12} in MS-II.

Table 3. Convergence of frequency parameters for angle-ply (30°/–30°/30°) SSSS square plates (MS-I)

Material	N	Mode numbers							
		1	2	3	4	5	6	7	8
E/E	11	15.8619	35.8018	42.5515	61.3169	71.6273	85.6521	93.5636	108.7378
	15	15.8621	35.8021	42.5519	61.3176	71.6287	85.6529	93.5625	108.7262
	19	15.8621	35.8021	42.5520	61.3177	71.6288	85.6529	93.5626	108.7263
	23	15.8622	35.8021	42.5520	61.3177	71.6289	85.6530	93.5626	108.7264
	27	15.8622	35.8021	42.5520	61.3177	71.6289	85.6530	93.5626	108.7264
	[3]	15.90	35.86	42.62	61.45	71.71	85.72	93.74	108.9
B/E	11	11.9625	22.4074	35.4364	37.4339	49.2075	55.9908	70.5661	73.0071
	15	11.9648	22.4100	35.4424	37.4329	49.2104	55.9665	70.4988	72.9975
	19	11.9655	22.4109	35.4444	37.4329	49.2123	55.9665	70.4998	72.9982
	23	11.9658	22.4112	35.4453	37.4329	49.2132	55.9665	70.5002	72.9986
	27	11.9659	22.4114	35.4457	37.4329	49.2136	55.9665	70.5005	72.9988
	[3]	12.21	22.78	35.86	37.90	50.04	56.70	71.36	73.57
G/E	11	11.6857	21.5346	35.4172	35.5276	48.6468	52.6563	69.1293	71.4666
	15	11.6894	21.5392	35.4255	35.5259	48.6519	52.6272	69.0619	71.4086
	19	11.6906	21.5407	35.4286	35.5259	48.6552	52.6272	69.0637	71.4088
	23	11.6911	21.5414	35.4299	35.5259	48.6569	52.6272	69.0645	71.4089
	27	11.6914	21.5417	35.4306	35.5259	48.6576	52.6272	69.0649	71.4090
	[3]	11.97	21.97	35.88	36.04	49.60	53.43	70.04	72.35

Table 4. Convergence of frequency parameters for angle-ply (45°/–45°/45°) SSSS square plates (MS-I)

Material	N	Mode numbers							
		1	2	3	4	5	6	7	8
E/E	11	16.0871	36.8624	41.7104	61.6715	76.9472	79.8778	94.4474	108.7482
	15	16.0876	36.8626	41.7116	61.6726	76.9474	79.8804	94.4454	108.7347
	19	16.0877	36.8627	41.7120	61.6728	76.9474	79.8810	94.4456	108.7352
	23	16.0878	36.8627	41.7121	61.6728	76.9474	79.8812	94.4456	108.7354
	27	16.0878	36.8627	41.7121	61.6728	76.9474	79.8813	94.4456	108.7355
	[3]	16.14	36.93	41.81	61.85	77.04	80.00	94.68	109.0
B/E	11	12.3054	24.1007	33.5834	39.5290	53.7288	58.3472	64.9806	76.8154
	15	12.3196	24.1000	33.6269	39.5290	53.7162	58.3201	65.0412	76.7477
	19	12.3253	24.0999	33.6443	39.5297	53.7162	58.3198	65.0666	76.7506
	23	12.3281	24.0999	33.6528	39.5301	53.7162	58.3197	65.0794	76.7522
	27	12.3298	24.0999	33.6576	39.5303	53.7162	58.3197	65.0868	76.7532
	[3]	12.71	24.51	34.44	40.23	54.44	59.40	66.38	78.00
G/E	11	11.8647	23.2991	33.2088	37.7016	53.3682	55.2007	64.6746	75.3103
	15	11.8774	23.2987	33.2480	37.7018	53.3533	55.1709	64.7192	75.2411
	19	11.8824	23.2987	33.2641	37.7028	53.3533	55.1707	64.7400	75.2449
	23	11.8850	23.2987	33.2720	37.7033	53.3533	55.1707	64.7507	75.2470
	27	11.8865	23.2987	33.2765	37.7036	53.3533	55.1707	64.7571	75.2482
	[3]	12.31	23.72	34.14	38.45	54.10	56.31	66.20	76.23

Table 5. Frequency parameters of angle-ply ($\theta/-\theta/\theta$) SSSS square plates (E/E, $N = 31$)

θ°	Methods	Mode numbers							
		1	2	3	4	5	6	7	8
0	DQM (I)	15.19467	33.29959	44.41877	60.77869	64.52979	90.30141	93.66415	108.5563
	Exact (I)	15.19467	33.29959	44.41877	60.77869	64.52979	90.30141	93.66415	108.5563
	DQM (II)	15.17055	33.24847	44.38711	60.68220	64.45675	90.14548	93.63063	108.4588
	Exact (II)	15.17055	33.24847	44.38711	60.68220	64.45675	90.14548	93.63063	108.4588
15	DQM (I)	15.4150	34.0748	43.8514	60.8068	66.6413	91.3847	91.5001	108.8889
	[3]	15.43	34.09	43.87	60.85	66.67	91.40	91.56	108.9
	DQM (II)	15.3959	34.0299	43.8199	60.7327	66.5601	91.3403	91.3773	108.7845
30	DQM (I)	15.8622	35.8021	42.5521	61.3177	71.6289	85.6530	93.5627	108.7265
	[3]	15.90	35.86	42.62	61.45	71.71	85.72	93.74	108.9
	DQM (II)	15.8534	35.7679	42.5238	61.2745	71.5463	85.5891	93.4889	108.6531
45	DQM (I)	16.0880	36.8627	41.7122	61.6729	76.9474	79.8813	94.4456	108.7356
	[3]	16.14	36.93	41.81	61.85	77.04	80.00	94.68	109.0
	DQM (II)	16.0842	36.8321	41.6880	61.6430	76.8622	79.8129	94.3878	108.6515

Table 6. Frequency parameters of angle-ply ($\theta/-\theta/\theta$) SSSS square plates (B/E, $N = 31$)

θ°	Methods	Mode numbers							
		1	2	3	4	5	6	7	8
0	DQM (I)	11.03935	17.36394	30.90502	40.37093	44.15742	51.12759	53.26851	69.45577
	Exact (I)	11.03935	17.36394	30.90502	40.37093	44.15742	51.12759	53.26851	69.45577
	DQM (II)	11.04440	17.37677	30.92123	40.37645	44.17759	51.14502	53.30614	69.50708
	Exact (II)	11.04440	17.37677	30.92123	40.37645	44.17759	51.14502	53.30614	69.50708
15	DQM (I)	11.3047	19.0789	33.1642	38.7790	45.2024	51.9267	59.1244	72.3957
	[3]	11.37	19.21	33.32	38.86	45.46	52.14	59.48	72.77
	DQM (II)	11.3089	19.0890	33.1790	38.7854	45.2210	51.9463	59.1566	72.4253
30	DQM (I)	11.9660	22.4115	35.4460	37.4329	49.2139	55.9665	70.5006	72.9989
	[3]	12.21	22.78	35.86	37.90	50.04	56.70	71.36	73.57
	DQM (II)	11.9678	22.4180	35.4527	37.4446	49.2295	55.9816	70.5315	73.0123
45	DQM (I)	12.3308	24.0999	33.6606	39.5305	53.7162	58.3197	65.0916	76.7538
	[3]	12.71	24.51	34.44	40.23	54.44	59.40	66.38	78.00
	DQM (II)	12.3315	24.1065	33.6659	39.5399	53.7361	58.3325	65.1028	76.7794

Table 7. Frequency parameters of angle-ply ($\theta/-\theta/\theta$) SSSS square plates (G/E, $N = 31$)

θ°	Methods	Mode							
		1	2	3	4	5	6	7	8
0	DQM (I)	11.28972	17.13178	28.69169	40.74023	45.15887	45.78291	54.08234	68.14209
	Exact (I)	11.28972	17.13178	28.69169	40.74023	45.15887	45.78291	54.08234	68.14209
	DQM (II)	11.28718	17.12536	28.68364	40.73740	45.14874	45.77484	54.06362	68.13483
	Exact (II)	11.28718	17.12536	28.68364	40.73740	45.14874	45.77484	54.06362	68.13483
15	DQM (I)	11.3927	18.5447	31.0178	39.0659	45.6444	47.9654	58.2869	67.4789
	[3]	11.46	18.69	31.20	39.15	45.91	48.19	58.70	67.84
	DQM (II)	11.3906	18.5398	31.0109	39.0628	45.6354	47.9567	58.2711	67.4670
30	DQM (I)	11.6916	21.5420	35.4311	35.5259	48.6582	52.6273	69.0651	71.4091
	[3]	11.97	21.97	35.88	36.04	49.60	53.43	70.04	72.35
	DQM (II)	11.6908	21.5389	35.4278	35.5204	48.6507	52.6203	69.0502	71.4019
45	DQM (I)	11.8875	23.2987	33.2794	37.7038	53.3533	55.1707	64.7612	75.2490
	[3]	12.31	23.72	34.14	38.45	54.10	56.31	66.20	76.63
	DQM (II)	11.8873	23.2956	33.2768	37.6996	53.3432	55.1652	64.7556	75.2369

4. Conclusions

The free vibration of mid-plane symmetric angle-ply laminated composite square plates with all edges simply supported is successfully solved by using the modified differential quadrature method (modified DQM). Three material systems are considered. The rate of convergence of the modified DQM is investigated. The results are tabulated for references.

Based on the results reported herein, one may conclude that the DQ data are highly accurate and can be served as the benchmark solutions. The difference in solutions of the mid-plane symmetric angle-ply laminated composite plates with two sets of equivalent material constants is clearly seen and thus care should be taken when highly accurate results are needed for comparisons in testing newly developed numerical methods. However, the difference is small and negligible from the practical point of view.

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