Behavior of frequencies of orthotropic rectangular plate with circular variations in thickness and density

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Abstract. In this article, authors dealt with general solution of differential equation of orthotropic rectangular plate with clamped boundary conditions under bi-parabolic temperature variations. Rayleigh Ritz technique is adopted to solve the resultant equation and evaluate the frequencies for first four modes of vibration. The effect of circular thickness and density on frequencies of orthotropic plate are analyzed which is not done yet. The objective of the study is to optimize the frequency modes by choosing the appropriate variation in plate parameters. Th findings of the study complete the objective of the article. All the results are provided in tabular form.

Keywords: orthotropic rectangular plate, thermal gradient, circular tapering, circular density.

1. Introduction

Orthotropic rectangular plate with various plate parameters is widely used in various engineering branches like aerospace, aircraft, automobiles etc. A significant work has been reported by many researchers on orthotropic rectangular plate along with different variations in plate parameters but few work has been reported to study the effect of circular variation in plate parameters (thickness and density).

Effect of nonhomogeneity on vibration of orthotropic rectangular plates of varying thickness resting on Pasternak foundation is discussed in [1]. Free vibration analysis of laminated composite plate with elastic point and line supports by using finite element method was presented by [2]. Buckling analysis of rectangular isotropic plate having simply supported boundary conditions under the influence of non-uniform in-plane loading by using first-order shear deformation theory (FSDT) has been implemented in [3] Free vibration of orthotropic rectangular plate with thickness and temperature variation is studied in [4] and [5] by using Rayleigh Ritz method. Theoretical analysis on time period of vibration of rectangular plate with different plate parameters was proposed by [6] and [7].

New straight forward benchmark solutions for bending and free vibration of clamped anisotropic rectangular thin plates by a double finite integral transform method is described in [8]. Natural vibration of skew plate on different set of boundary conditions with temperature gradient is analyzed in [9]. Free vibration analysis of isotropic and laminated composite plate [10] on elastic point supports using finite element method (FEM) is studied. Vibration frequencies of a rectangular plate with linear variation in thickness and circular variation in Poisson's ratio was provided in [11]. Thermally induced vibrations of shallow functionally graded material arches is considered and analyzed in [12] by using classical theory of curved beams. Fast converging semi analytical method was developed by [13] for assessing the vibration effect on thin orthotropic skew (or parallelogram/oblique) plates. A mathematical model is presented in [14] to analyse the vibration of a tapered isotropic rectangular plate under thermal condition.

In this study, authors examined the behavior of frequencies of orthotropic rectangular plate having circular variations in thickness and density. Authors also discussed the frequency variation when temperature increased on the plate. The present study provided how we can optimize the frequency value by choosing circular variation in plate parameters which will be helpful to structural engineering to get ride from unwanted vibrations.

2. Problem geometry and analysis

Consider a nonhomogeneous orthotropic rectangle plate with sides a and b, thickness l and density ρ as shown in Fig. (1).

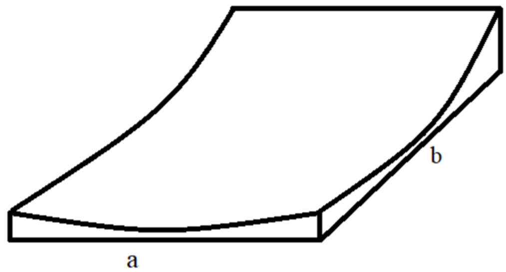

Fig. 1. Orthotropic rectangular plate with $2D$ circular thickness

The kinetic energy and strain energy for vibration of plate are expressed in the following manner as in [15]:

$$T_s = \frac{1}{2}\omega^2 \int_0^a \int_0^b \rho l \Phi^2 d\psi d\zeta, \tag{1}$$

$$V_s = \frac{1}{2}\int_0^a \int_0^b \left[[D_x \left(\frac{\partial^2 \Phi}{\partial \zeta^2}\right)^2 + D_y \left(\frac{\partial^2 \Phi}{\partial \psi^2}\right)^2 + 2\nu_x D_y \frac{\partial^2 \Phi}{\partial \zeta^2}\frac{\partial^2 \Phi}{\partial \psi^2} + 4D_{xy}\left(\frac{\partial^2 \Phi}{\partial \zeta \partial \psi}\right)^2\right] d\psi d\zeta, \tag{2}$$

where Φ deflection function, ω natural frequency, $D_x = E_x l^3/12(1-\nu_x\nu_y)$, $D_y = E_y l^3/12(1-\nu_x\nu_y)$ are flexural rigidities in x and y directions respectively and $D_{xy} = E_x l^3/12(1-\nu_x\nu_y)$ is torsional rigidity.

To solve frequency equation, Rayleigh Ritz technique is implemented which requires:

$$L = \delta(V_s - T_s) = 0. \tag{3}$$

Using Eqs. (1), (2), we get functional equation:

$$\begin{aligned} L = \frac{1}{2}\int_0^a \int_0^b \left[D_x \left(\frac{\partial^2 \Phi}{\partial \zeta^2}\right)^2 + D_y \left(\frac{\partial^2 \Phi}{\partial \psi^2}\right)^2 + 2\nu_x D_y \frac{\partial^2 \Phi}{\partial \zeta^2}\frac{\partial^2 \Phi}{\partial \psi^2} + 4D_{xy}\left(\frac{\partial^2 \Phi}{\partial \zeta \partial \psi}\right)^2\right] d\psi d\zeta \\ -\frac{1}{2}\omega^2 \int_0^a \int_0^b \rho l \Phi^2 d\psi d\zeta = 0. \end{aligned} \tag{4}$$

Now, introducing non-dimensional variable as:

$$\zeta_1 = \frac{\zeta}{a}, \quad \psi_1 = \frac{\psi}{a},$$

together with two dimensional circular thickness, we get:

$$l = l_0\left(1 + \beta_1\left\{1 - \sqrt{1 - {\zeta_1}^2}\right\}\right)\left(1 + \beta_2\left\{1 - \sqrt{1 - \frac{a^2}{b^2}{\psi_1}^2}\right\}\right), \tag{5}$$

where l_0 is thickness at origin and β_1, β_2 $(0 \leq \beta_1, \beta_2 \leq 1)$ are tapering parameters.

For non-homogeneity consideration, authors assumed one dimensional circular variation in density:

$$\rho = \rho_0 \left(1 + m_0 \left\{1 - \sqrt{1 - {\zeta_1}^2}\right\}\right). \tag{6}$$

Two dimensional parabolic temperature distribution is taken as in [5]:

$$\tau = \tau_0 (1 - \zeta_1^2) \left(1 - \frac{a^2}{b^2} {\psi_1}^2\right), \tag{7}$$

where τ and τ_0 denotes the temperature on and at the origin respectively.

For orthotropic materials, modulus of elasticity is given by [17]:

$$E_x = E_1 (1 - \gamma\tau), \quad E_y = E_2 (1 - \gamma\tau), \quad G_{xy} = G_0 (1 - \gamma\tau), \tag{8}$$

where E_x and E_y are the Young's modulus in x and y directions, G_{xy} is shear modulus and γ is called slope of variation. Using Eq. (7), Eq. (8) becomes:

$$\begin{aligned} E_x(x) &= E_1 \left(1 - \alpha (1 - \zeta_1^2) \left(1 - \frac{a^2}{b^2} {\psi_1}^2\right)\right), \\ E_y(x) &= E_2 \left(1 - \alpha (1 - \zeta_1^2) \left(1 - \frac{a^2}{b^2} \psi_1^{\ 2}\right)\right), \\ G_{xy(x)} &= G_0 \left(1 - \alpha (1 - \zeta_1^2) \left(1 - \frac{a^2}{b^2} {\psi_1}^2\right)\right), \end{aligned} \tag{9}$$

where $\alpha = \gamma\tau_0$ $(0 \leq \alpha < 1)$ is called thermal gradient. Using Eqs. (5), (6), (9) and non dimensional variable, the functional in Eq. (4) become:

$$\begin{aligned} L = \frac{D_0}{2} \int_0^1 \int_0^{\frac{b}{a}} \Bigg[& \left(1 - \alpha (1 - \zeta_1^2) \left(1 - \frac{a^2}{b^2} {\psi_1}^2\right)\right) \left(1 + \beta_1 \left\{1 - \sqrt{1 - {\zeta_1}^2}\right\}\right)^3 \Bigg(1 \\ & + \beta_2 \left\{- \sqrt{1 - \frac{a^2}{b^2} {\psi_1}^2}\right\}\Bigg)^3 \left\{ \left(\frac{\partial^2 \Phi}{\partial \zeta_1^2}\right)^2 + \frac{E_2}{E_1} \left(\frac{\partial^2 \Phi}{\partial \psi_1^2}\right)^2 + 2\nu_x \frac{E_2}{E_1} \left(\frac{\partial^2 \Phi}{\partial \zeta_1^2}\right) \left(\frac{\partial^2 \Phi}{\partial \psi_1^2}\right) \right. \\ & \left. + 4 \frac{G_0}{E_1} (1 - \nu_x \nu_y) \left(\frac{\partial^2 \Phi}{\partial \zeta_1 \partial \psi_1}\right)^2 \right\} \Bigg] d\psi_1 d\zeta_1 - \lambda^2 \int_0^1 \int_0^{\frac{b}{a}} \left[\left(1 + m_0 \left\{1 - \sqrt{1 - {\zeta_1}^2}\right\}\right) \Bigg(1 \right. \\ & \left. + \beta_2 \left\{1 - \sqrt{1 - {\zeta_1}^2}\right\}\Bigg) \left(1 + \beta_2 \left\{1 - \sqrt{1 - \frac{a^2}{b^2} \psi_1^{\ 2}}\right\}\right) \right] \Phi^2 d\psi_1 d\zeta_1 = 0, \end{aligned} \tag{10}$$

where:

$$D_0 = \frac{1}{2} \left(\frac{E_1 l_0^3}{12(1 - \nu_x \nu_y)}\right), \quad \lambda^2 = \frac{12 a^4 \rho \omega^2 (1 - \nu_x \nu_y)}{E_1 {h_0}^2}.$$

We choose deflection function which satisfies all the edge condition as taken in [16]:

$$\Phi(\zeta,\psi) = \left[(\zeta_1)^e(\psi_1)^f(1-\zeta_1)^g\left(1-\frac{a\psi_1}{b}\right)^h\right]\left[\sum_{i=0}^{n}\Psi_i\left\{(\zeta_1)(\psi_1)(1-\zeta_1)\left(1-\frac{a\psi_1}{b}\right)\right\}^i\right], \quad (11)$$

where $\Psi_i, i = 0,1,2\ldots n$ are unknowns and the value of e, f, g, h can be 0, 1 and 2, corresponding to free, simply supported and clamped edge conditions respectively. To minimize Eq. (11), we impose the following condition:

$$\frac{\partial L}{\partial \Psi_i} = 0, \quad i = 0,1,\ldots n. \quad (12)$$

Solving Eq. (11), we have the following frequency equation:

$$|P - \lambda^2 Q| = 0, \quad (13)$$

where $P = [p_{ij}]_{i,j=0,1,..n}$ and $Q = [q_{ij}]_{i,j=0,1,..n}$ are square matrix of order $(n+1)$.

3. Numerical results and discussion

Vibrational frequency of orthotropic rectangular plate having circular variation for first four modes under thermal effect with circular variation in thickness and density is discussed. Frequency equation is solved by using Rayleigh-Ritz technique with the help of Maple software. Frequency values for first four modes is calculated on different variations of plate parameters (i.e., thermal gradient, tapering parameters and nonhomogeneity). The results have been presented in the form of tables. In through out calculation the aspect ratio i.e., ratio of length to breadth of the plate is considered to be 1.5.

Tables 1 and 2 represents the frequencies for first four modes corresponding to tapering parameters β_1, β_2 for three different values of thermal gradient, tapering paramter and nonhomogeneity m i.e., $\alpha = \beta_2 = m = 0.2$, $\alpha = \beta_2 = m = 0.4$, $\alpha = \beta_2 = m = 0.6$. (In Table 1) i.e., $\alpha = \beta_1 = m = 0.2$, $\alpha = \beta_1 = m = 0.4$, $\alpha = \beta_1 = m = 0.6$ (In Table 2).

Table 1. Tapering parameter β_1 vs vibrational frequency λ

β_1	$\alpha = \beta_2 = m = 0.2$				$\alpha = \beta_2 = m = 0.4$				$\alpha = \beta_2 = m = 0.6$			
	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
0.0	16.2250	62.5266	140.527	314.224	15.4920	59.2450	132.974	303.002	114.7460	55.8753	125.414	291.617
0.2	16.9440	64.3744	145.547	331.268	16.2045	61.5334	137.954	319.324	15.4520	58.1060	130.414	306.785
0.4	17.7186	67.3744	151.897	347.316	16.9703	63.9622	143.549	334.372	16.2086	60.4747	135.793	322.357
0.6	18.5423	70.0079	156.897	366.204	17.7826	66.5125	149.182	353.426	17.0089	62.9527	141.414	341.070
0.8	19.4086	72.7501	163.162	384.709	18.6352	69.1850	155.236	372.291	17.8470	65.5327	147.325	359.995
1.0	20.3118	75.5910	169.577	405.599	19.5224	71.9365	161.609	391.622	18.7174	68.1935	153.484	380.389

Table 2. Tapering parameter β_2 vs vibrational frequency λ

β_2	$\alpha = \beta_1 = m = 0.2$				$\alpha = \beta_1 = m = 0.4$				$\alpha = \beta_1 = m = 0.6$			
	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
0.0	16.6318	63.8209	143.168	325.058	16.3257	61.8337	138.677	320.118	16.0163	59.7280	134.016	317.501
0.2	16.9440	64.8730	145.547	331.223	16.6431	62.8843	141.117	326.682	16.3380	60.7803	136.374	325.163
0.4	17.2665	65.9491	147.895	339.639	16.9703	63.9622	143.549	334.372	16.6691	61.8582	138.930	332.343
0.6	17.5987	67.0584	150.517	345.112	17.3068	65.0711	146.010	342.215	17.0089	62.9517	141.463	340.544
0.8	17.9401	68.1874	153.129	352.618	17.6521	66.2017	148.587	349.894	17.3571	64.0756	144.056	348.595
1.0	18.2902	69.3457	155.639	361.626	18.0057	67.3523	151.159	359.151	17.7131	65.2175	146.720	357.118

From Tables 1 and 2, the following facts can be interpreted:

1) Frequency increases in both the Tables 1 and 2 with the increasing in value of tapering

parameters β_1 and β_2.

2) for fixed values of both the tapering parameters β_1 and β_2, frequency decreases in both the Tables 1 and 2.

3) Rate of decrement of frequencies reported in Table 2 is much slow as compared to Table 1.

Table 3 summarizes the frequency for first four modes corresponding to thermal gradient α for three different value of tapering parameters β_1, β_2 and non homogeneity m i.e., $\beta_1 = \beta_2 = m = 0.2$, $\beta_1 = \beta_2 = m = 0.4$, $\beta_1 = \beta_2 = m = 0.6$. Table 4 presents frequency for first four modes corresponding to nonhomogeneity m for three different values of thermal gradient α, tapering paramters β_1, β_2 i.e., $\beta_1 = \beta_2 = \alpha = 0.2$, $\beta_1 = \beta_2 = \alpha = 0.4$, $\beta_1 = \beta_2 = \alpha = 0.6$ respectively.

From Tables 3 and 4, it can be concluded that:

1) Increase in thermal gradient α and nonhomogeneity m results in decrease in frequency in both the tables 3 and 4.

2) Rate of decrement of frequency in nonhomogeneity is slower as compared to thermal gradient.

3) Here, behavior of both the Tables 3 and 4 is quite similar but in Table 4 have higher frequency values as compared to Table 3.

4) Without thermal effect, Table 3 have maximum frequency value then it decreases gradually with the increase in thermal gradient.

Table 3. Thermal gradient α vs vibrational frequency λ

α	$\beta_1 = \beta_2 = m = 0.2$				$\beta_1 = \beta_2 = m = 0.4$				$\beta_1 = \beta_2 = m = 0.6$			
	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
0.0	17.7264	68.0981	152.773	344.414	18.5608	70.5842	158.161	359.868	19.4501	73.1977	163.831	377.364
0.2	16.9440	64.8730	145.547	331.268	17.7860	67.3676	151.009	347.406	18.6797	69.9838	156.690	365.602
0.4	16.1196	61.4568	138.000	317.463	16.9703	63.9628	143.544	334.381	17.8688	66.5801	149.261	353.332
0.6	15.2450	57.8227	129.915	303.204	16.1054	60.3435	135.542	321.048	17.0089	62.9563	141.409	340.651
0.8	14.3085	53.9052	121.350	288.101	15.1795	56.4381	127.131	306.930	16.0884	59.0494	133.108	327.429
1.0	13.2923	49.6287	112.057	272.273	14.1752	52.1774	118.015	292.304	15.0893	54.7852	124.157	313.803

Table 4. Nonhomogeneity m vs vibrational frequency λ

m	$\alpha = \beta_1 = \beta_2 = 0.2$				$\alpha = \beta_1 = \beta_2 = 0.4$				$\alpha = \beta_1 = \beta_2 = 0.6$			
	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
0.0	17.2013	65.9667	148.314	335.442	17.4856	66.1290	148.314	348.144	17.7821	66.1528	149.118	362.262
0.2	16.9440	64.8663	145.634	331.105	17.2222	65.0210	145.943	341.903	17.5128	65.0309	146.362	355.401
0.4	16.6979	63.8335	143.210	323.724	16.9703	63.9628	143.544	334.381	17.2554	63.9650	143.309	348.519
0.6	16.4622	62.8448	140.797	318.704	16.7292	62.9605	141.119	328.422	17.0089	62.9572	141.309	341.897
0.8	16.2363	61.8978	138.611	313.065	16.4980	62.0028	138.814	323.043	16.7728	61.9888	139.107	334.293
1.0	16.0193	60.9914	136.613	306.693	16.2762	61.0890	136.707	316.661	16.5461	61.0639	136.860	329.374

4. Conclusions

Present model provides frequencies of orthotropic rectangular plate with circular variations in thickness and density. from above results and discussion, authors concluded that increasing in tapering parameters β_1 and β_2, results the decrease in frequency as shown in tables 1 and 2. but increase in thermal gradient α and non homogeneity m, results the increase in frequency is illustrated in tables 3 and 4. the variation in frequency mode (increasing or decreasing) are very slow due to implementation of circular variation in plate parameters i.e., there is no sudden increment or decrement reported in frequencies.

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