Production analysis of manufacturing industry in a single vacation policy under disaster

Jeyakumar S¹, Logapriya B²

¹Department of Mathematics, Government Arts College, Coimbatore, India
²Department of Science and Humanities, Sri Krishna College of Engineering and Technology, Coimbatore, India

Corresponding author
E-mail: ¹jeyakumar_19@yahoo.co.in, ²logapriyab@skcet.ac.in

Received 27 October 2023; accepted 11 November 2023; published online 22 November 2023

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Abstract. The disaster in queueing system with second optional service is considered. Arriving customer of this system will receive the essential service and optional second service if needed. When the system is interrupted by the disaster, the server initiates the repair period making all the customer leave the system immediately. The server, when idle, takes single vacation. The disaster cannot happen when server is under vacation or in repair period. The above queueing system is analysed using supplementary variable technique to obtain the probability generating function for various parameters and effects of parameters are explained graphically with numerical illustrations.

Keywords: supplementary variable technique, second optional service, disaster, repairs, single vacation.

1. Introduction

The disaster otherwise named as catastrophes can be seen in manufacturing systems, communication systems and many more. As disaster removes all customers immediately from the system, it is extensively modelled by many researchers.


The literature study of the researchers in Markovian queue with disaster under different parameter motivated to study the effect of disaster in Markovian queue with optional service under single vacation policy. There are works in bulk arrival, service under disaster and queues with optional service, but on motivation from literature study, novelty in the described model is linking bulk arrival with optional service and disaster to evaluate the performance measures in arbitrary epoch.

Importance and contribution of study is to find the disaster rate affecting the production, to evaluate and analyze the performance measures of the queuing model with numerical illustrations to establish results in the real word existing scenario.

The queue with optional service under disaster and repairs with single vacation can be identified in ice-cream manufacturing unit. When raw material reaches the unit, it is immediately sent to mixer which is essential service and adding up flavour is a manual process which is optional service. Once it is mixed up, it is poured into mould and kept in dipping freezer. When sudden freezer burn (disaster) happens in freezer, the entire process has to stop since the mixed-up raw material can’t be kept for long period. Before starting up next slot of opening the raw material for mix-up, the server can take single vacation if needed because raw material should be fresh for good taste and flavour.

2. Mathematical model

In a compound Poisson process, a batch of customers $i$, with the parameters $\lambda, \lambda > 0$ and probability $\lambda c_i dt$ joins the system in the minimal duration of time $(t, t + dt)$. Each and every customer who arrives at the system is provided with first essential service and optional service. Customer is served one by one in the batch. Let $\mu_1(x) = \frac{S_1'(x)}{1-S_1(x)}$ be the hazard rate function of the first essential service with $S_1'(s) = \int_0^s e^{-sx} dS_1(x)$, where $S_1$ is the general distribution function with the corresponding density function $S_1'$ and mean $E(S_1)$. After the first service is completed, the optional service may be opted with the probability ‘$s$’ or leave the system with the probability $1-s$. For optional service, let $\mu_2(x) = \frac{S_2'(x)}{1-S_2(x)}$ with $S_2'(s) = \int_0^s e^{-sx} dS_2(x)$, where $S_2$ is the general distribution function with the corresponding density function $S_2'$ and mean $E(S_2)$. Once the server completes his essential and optional service to the customer, the server may take vacation when he is idle. Let $\nu(x) = \frac{L'(x)}{1-L(x)}$ be the rate function of the vacation with $L'(s) = \int_0^s e^{-sx} dL(x)$, where $L$ is the general distribution function with the corresponding density function $L'$ and mean $E(L)$. Finally, disaster is assumed to happen in the system during essential service, since optional service is the manual service. With rate of the disaster $\delta$, it removes all the customers including one is been served from the system and the system is immediately moved to repair period. Let $r(x) = \frac{R'(x)}{1-R(x)}$ be the hazard rate function of the repair period with $R'(s) = \int_0^s e^{-sx} dR(x)$, where $R$ is the general distribution function with density function $R'$ and mean $E(R)$. 
3. Notations and abbreviations

At time $t$, let the system size be $N_t$. Also, let $\Omega(t)$ be the random variable corresponding to server’s status.

Let $S_i(t)$, $i = 1, 2$ and $L_t$ be introduced as supplementary variables to obtain a Markov process. Limiting probabilities are defined to derive Kolmogorov chapman equation as:

$$P_0 = \lim_{t \to \infty} \{N_t = 0, \Omega(t) = 0\},$$

$$P_n^{(e)}(x) = \lim_{t \to \infty} \{N_t = n, \Omega(t) = 1, x < S_{1,t} < x + dt\}, \quad n \geq 1,$$

$$P_n^{(o)}(x) = \lim_{t \to \infty} \{N_t = n, \Omega(t) = 2, x < S_{2,t} < x + dt\}, \quad n \geq 1,$$

$$R_n(x) = \lim_{t \to \infty} \{N_t = n, \Omega(t) = 3, x < R_t < x + dt\}, \quad n \geq 1,$$

$$L_n(x) = \lim_{t \to \infty} \{N_t = n, \Omega(t) = 4, x < L_t < x + dt\}, \quad n \geq 1.$$

4. Steady state differential equations using supplementary variable technique

Governing equations for various states of the system are framed for $n > 0$:

$$0 = -\lambda P_0 + \int_0^\infty L_0(x) v(x) dx$$

$$+ \left( \int_0^\infty P_1^{(o)}(x) \mu_2(x) dx + (1 - s) \int_0^\infty P_0^{(e)}(x) \mu_3(x) dx + \int_0^\infty R_0(x) r(x) dx \right),$$

$$\left( \frac{d}{dx} + \lambda + \mu_1(x) + \delta \right) P_n^{(e)}(x) = \lambda \sum_{i=1}^{n-1} C_i P_{n-i}^{(e)}(x), \quad n \geq 1,$$

$$\left( \frac{d}{dx} + \lambda + \mu_2(x) \right) P_n^{(o)}(x) = \lambda \sum_{i=1}^{n-1} C_i P_{n-i}^{(o)}(x), \quad n \geq 1,$$

$$\left( \frac{d}{dx} + \lambda + r(x) \right) R_n(x) = \lambda \sum_{i=1}^{n-1} C_i R_{n-i}^{(o)}(x), \quad n \geq 1,$$

$$\left( \frac{d}{dx} + \lambda + r(x) \right) R_0(x) = 0,$$

$$\left( \frac{d}{dx} + \lambda + v(x) \right) L_n(x) = \lambda \sum_{i=1}^{n-1} C_i L_{n-i}(x), \quad n \geq 1.$$
\( \frac{d}{dx} + \lambda + v(x) \) \( L_0(x) = 0. \) 

With the boundary conditions:

\[ P_n^{(e)}(0) = \int_0^\infty L_n(x)v(x)dx + \int_0^\infty p_n^{(e)}(x)\mu_2(x)dx \]

\[ + (1-s) \int_0^\infty p_{n+1}(x)\mu_1(x)dx + \int_0^\infty R_n(x)r(x)dx + \lambda C_n P_0, \quad n \geq 0, \]

\[ P_n^{(o)}(0) = s \int_0^\infty p_n^{(e)}(x)\mu_1(x)dx, \]

\[ R_0(0) = \delta \sum_{n=1}^\infty \int_0^\infty p_n^{(e)}(x)dx, \]

\[ L_n(0) = \int_0^\infty p_n^{(o)}(x)\mu_2(x)dx + (1-s) \int_0^\infty p_n^{(e)}(x)\mu_1(x)dx. \]

To obtain PGF, we multiply the Eqs. (2-11) with certain powers of \( z \) and sum it over \( n \), to obtain:

\[ P^{(e)}(0, z) = \int_0^\infty (L(x,z) - L_0(x))v(x)dx + \int_0^\infty \left( \frac{1}{z} p^{(o)}(x,z) - p^{(e)}(x,z) \right) \mu_2(x)dx \]

\[ + (1-s) \int_0^\infty \left( \frac{1}{z} p^{(e)}(x,z) - p^{(e)}(x) \right) \mu_1(x)dx + \int_0^\infty (R(x,z) - R_0(x))r(x)dx + \lambda C(z) P_0, \]

\[ P^{(o)}(0, z) = s \int_0^\infty p^{(e)}(x,z)\mu_1(x)dx, \]

\[ R(0, z) = \delta \int_0^\infty p^{(e)}(x,1)dx, \]

\[ L(0, z) = \int_0^\infty p_n^{(o)}(x,z)\mu_2(x)dx + (1-s) \int_0^\infty p_n^{(e)}(x,z)\mu_1(x)dx. \]

As the system has no customer after immediate vacation and repair time that comes after disaster, we have:

\[ L(0, z) = L_0(0), \]

\[ R(0, z) = R_0(0). \]

5. PGF of queue for various states:

At an arbitrary epoch, the PGF of the server is under busy state, repair state and in vacation are obtained from the explicit expressions:
\[ P^{(e)}(0, z) = \frac{z \left( \delta P^{(e)}(1) R^*(A_z) - \frac{\lambda P_0}{\beta} H_z \right)}{z - S'_1(A_z + \delta) \left( (1 - s) + s S'_2(A_z) \right)} \]  

(16)

where \( A_z = \lambda - \lambda C(z) \) and \( H_z = \beta \left( 1 - C(z) \right) + \left( 1 - L^*(A_z) \right) \).

If possible, using Rouche’s theorem, let \( z = z_\theta \) be the unique solution of \( z = S'_1(A_z + \delta) \left( (1 - s) + S'_2(A_z) \right) \). In that case equality becomes, \( \frac{\lambda P_0}{\beta} H_z = \delta P^{(e)}(1) R^*(A_z) \).

The PGF of the system at epoch of essential service to the customer is given as:

\[ P^{(e)}(z) = \frac{z \left( \pi \frac{\lambda P_0}{\beta} R^*(A_z) - \frac{\lambda P_0}{\beta} H_z \right)}{z - S'_1(A_z + \delta) \left( (1 - s) + S'_2(A_z) \right) \left( \frac{1 - S'_1(A_z + \delta)}{A_z + \delta} \right)} \]  

(17)

The PGF of the system at epoch of optional second service to the customer is given as:

\[ P^{(o)}(z) = \frac{z \left( \pi \frac{\lambda P_0}{\beta} R^*(A_z) - \frac{\lambda P_0}{\beta} H_z \right)}{z - S'_1(A_z + \delta) \left( (1 - s) + S'_2(A_z) \right) \left( \frac{s(S'_1(A_z + \delta))(1 - S'_2(A_z))}{A_z} \right)} \]  

(18)

The PGF of the system at epoch under repair is given as:

\[ R(z) = \frac{\pi \lambda P_0}{\beta} \frac{1 - R^*(A_z)}{A_z} \]  

(19)

The PGF of the system at epoch of under vacation is given as:

\[ L(z) = \frac{\lambda P_0}{\beta} \frac{1 - L^*(A_z)}{A_z} \]  

(20)

Using the normalizing condition at \( z = 1 \), the probability \( P_0 \) is obtained when the server being idle as:

\[ P_0 = \left( 1 + \frac{\lambda}{\beta} \left( \frac{\pi S'_1(\delta) E(S_z)}{1 - S'_1(\delta)} + \pi E(R) + E(L) \right) \right)^{-1} \]  

(21)

Finally, the total PGF of queue size \( X(z) \) is obtained at as:

\[ X(z) = \left( 1 + \frac{\lambda}{\beta} \left( \frac{\pi s S'_1(\delta) E(S_z)}{1 - S'_1(\delta)} + \pi E(R) + E(L) \right) \right)^{-1} \left[ 1 + \frac{\lambda}{\beta} \frac{(1 - L^*(A_z))}{A_z} + \frac{\pi (1 - R^*(A_z))}{A_z} \right] \]

\[ + \frac{z(\pi R^*(A_z) - H_z)}{z - S'_1(A_z + \delta) \left( (1 - s) + S'_2(A_z) \right) \left( \frac{1 - S'_1(A_z + \delta)}{A_z + \delta} \right)} \]

\[ + \frac{s(S'_1(A_z + \delta))(1 - S'_2(A_z))}{A_z} \]  

(22)
Also, due to disaster the PGF of total customers removed from the system is given as:

$$X_d(z) = \frac{z\delta(\pi R^*(A_z) - H_z)}{\pi(z - S_1^*(A_z + \delta))(1 - s + s.S_2^*(A_z))} \left(1 - S_1^*(A_z + \delta)\right).$$

(23)

6. Performance measures

Expected queue length ($E(Q_L)$) and waiting time ($E(W_T)$) is obtained using L’ Hospital’s rule from the Eq. (26):

$$E(Q_L) = \frac{\lambda E(c)}{\beta} \left(1 + \frac{\lambda}{\beta} \left(\frac{\pi + s\pi S_1^*(\delta)E(S_2) + \pi E(R) + E(L)}{1 - S_1^*(\delta)}\right)\right)^{-1} \left(S_b + S_{b\delta}\right)$$

$$+ \frac{1}{S_c} \left(\pi(1 - S_c) \left(-\frac{1}{\delta E(c)} - \frac{\lambda s}{2} S''(0)\right) + S_b s(1 - S_c)E(S_2)\right)$$

$$- \frac{s\pi E(S_2)}{S_c} \left(\left(1 - S_c\right)^2 + \lambda S_1^*(\delta)\right).$$

(24)

where $S_c = 1 - S_1(\delta)$, $S_b = \beta + \frac{\lambda \pi}{\delta} + \lambda \pi E(R) + \lambda E(L)$, $S_{b\delta} = \frac{\lambda}{2} \left(\pi E(R^2) + E(L^2)\right)$.

Let $S_2''(0)$, $R''(0)$, $L''(0)$ are the mean service time of all the states; $S_2''''(0)$, $R''''(0)$, $L''''(0)$ be the second moment of states in different epoch. Using little’s formula, the expected waiting time is calculated as:

$$E(W_T) = \left(1 + \frac{\lambda}{\beta} \left(\frac{\pi + s\pi S_1^*(\delta)E(S_2) + \pi E(R) + E(L)}{1 - S_1^*(\delta)}\right)\right)^{-1} \left(S_b + S_{b\delta}\right)$$

$$+ \frac{1}{S_c} \left(\pi(1 - S_c) \left(-\frac{1}{\delta E(c)} - \frac{\lambda s}{2} S''(0)\right) + S_b s(1 - S_c)E(S_2)\right)$$

$$- \frac{s\pi E(S_2)}{S_c} \left(\left(1 - S_c\right)^2 + \lambda S_1^*(\delta)\right).$$

(25)

7. Numerical illustration

A numerical approach is carried on to evaluate the various states of the distribution. $\mu$, $r$, $v$ are assumed to be in exponential parameter respectively. Arrival rate is $\lambda$ with service rate $\mu_1 = 5$ and $\mu_2 = 6$. Whereas disaster may take place with rate $\delta$.

Average queue length and mean waiting time are computed and tabulated below with assumption of arrival and vacation as single.

From Table 1 and Fig. 2 it is understood that when probability of second optional service increases, length of the queue and waiting time to be increased. From Table 2 and Fig. 3, queue length and waiting time decreases as the disaster rate increases. Finally, from Table 3 and Fig. 4 when arrival rate increases then length of the queue and waiting time of the server increases.
Table 1. Performance measures with $\lambda = 2$ and $\delta = 2$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$P_0$</th>
<th>$E(Q_L)$</th>
<th>$E(W_T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.6521</td>
<td>1.8464</td>
<td>0.9232</td>
</tr>
<tr>
<td>0.25</td>
<td>0.6401</td>
<td>1.9829</td>
<td>0.9915</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6285</td>
<td>2.1146</td>
<td>1.0573</td>
</tr>
<tr>
<td>0.75</td>
<td>0.6173</td>
<td>2.2415</td>
<td>1.1207</td>
</tr>
<tr>
<td>1</td>
<td>0.6065</td>
<td>2.3640</td>
<td>1.1820</td>
</tr>
</tbody>
</table>

Fig. 2. Prob. of second optional service versus queue length and waiting time

Table 2. Performance measures with $\lambda = 2$ and $s = 1$

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$P_0$</th>
<th>$E(Q_L)$</th>
<th>$E(W_T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.6065</td>
<td>2.3640</td>
<td>1.1820</td>
</tr>
<tr>
<td>3</td>
<td>0.6393</td>
<td>2.3367</td>
<td>1.1683</td>
</tr>
<tr>
<td>4</td>
<td>0.6571</td>
<td>2.2998</td>
<td>1.1499</td>
</tr>
<tr>
<td>5</td>
<td>0.6682</td>
<td>2.2695</td>
<td>1.1347</td>
</tr>
<tr>
<td>6</td>
<td>0.6758</td>
<td>2.2456</td>
<td>1.2280</td>
</tr>
</tbody>
</table>

Fig. 3. Effect of disaster on queue length

Table 3. Performance measures with $s = 1$ and $\delta = 5$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$P_0$</th>
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<th>$E(W_T)$</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>0.8210</td>
<td>1.1333</td>
<td>1.1333</td>
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<tr>
<td>1.2</td>
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<tr>
<td>1.4</td>
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<td>1.5878</td>
<td>1.1341</td>
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<tr>
<td>1.6</td>
<td>0.7260</td>
<td>1.8151</td>
<td>1.1344</td>
</tr>
<tr>
<td>1.8</td>
<td>0.6965</td>
<td>2.0423</td>
<td>1.1346</td>
</tr>
</tbody>
</table>
8. Conclusions

In this Paper, $M^X/G/1$ Queue with the second optional service with single vacation is taken into consideration for disaster and repairs. We derive the queue size distributions for various steady-states with quality metrics and special cases using the supplementary variable technique. Additionally, we've provided numerical examples to illustrate the approach, which amply demonstrate how disasters affect waiting times and queue length with the second optional service. From the numerical illustration, it is observed that when arrival rate and probability for second optional service increases, queue length and waiting time of the customer increases as well as when disaster rate increases queue length and of the server decreases and waiting time of the customer decreases.

Acknowledgements

The authors have not disclosed any funding.

Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Author contributions

Jeyakumar S: project administration, supervision, validation. Logapriya B: conceptualization, formal analysis, investigation, methodology, visualization, writing – review and editing.

Conflict of interest

The authors declare that they have no conflict of interest.

References


Fig. 4. Effect of arrival rate and queue length


Jeyakumar S is currently working as an Assistant Professor in Department of Mathematics, Government Arts College, Coimbatore, India. He received his M.Sc and M.Phil in Mathematics from Bharathiar University, Coimbatore, India in 1988 and 1996 respectively, PGDCA from Bharathiar University in 1989 and Ph.D. from Bharathiar University in 2007. His research interests include queueing theory, graph theory and computer applications. He has publications in various journals which includes Applied Mathematical Modelling, Applied Mathematics and Computations, OPSEARCH, International journal of Operational Research etc.

Logapriya B is currently working as an Assistant Professor in Department of Science and Humanities in Sri Krishna College of Engineering and Technology, Coimbatore, India. She received her M.Sc and M.Phil in Mathematics from Bharathiar University, Coimbatore India in 2008 and 2014 respectively. Her research interest includes queueing theory, fluid dynamics and graph theory. She has publications in Pure and Applied Mathematics.