Stodola-Vianello iteration method for the free flexural vibration frequencies of Shimpi's single variable shear deformable beams

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Abstract. The natural vibration frequency analysis of beams is vital for their design against resonance failures because such failures occur when the excitation load frequencies of vibration coincide with such natural frequencies. This work presents a single variable shear deformable beam equation formulated using Shimpi's displacement field assumptions. This results in a quadratic shear stress profile over the depth and a satisfaction of the transverse shear stress-free boundary conditions. The governing equation is obtained using a first principles consideration and equilibrium method as a partial differential equation (PDE) which is non-homogenous for forced vibrations and homogeneous for free vibrations. The study then used the Stodola-Vianello iteration method to solve the resulting homogeneous PDE for simply supported boundary conditions and harmonic response. The problem reduced to an iterative problem of algebra involving the computation of an $(n + 1)$th vibratory modal shape function from an nth shape function that satisfies the boundary conditions. This work used a sinusoidal shape function which is exact for the simply supported boundary condition investigated. The use of boundary conditions solved the integration constants involved. Application of the convergence rule led to the eigenequation from which the eigenvalues were found. The eigenvalues were presented for the first four modes of vibration and for a rectangular beam. It was found that for l/h varying from 5 to 100, the natural vibration frequencies were identical with the ω_n values obtained using Navier method for other thick beam vibration problems. It was also found that ω_n was close to the exact values for all vibration modes and for all values of l/h between 5 and 100. For all vibration modes and all considered l/h values negligible differences, were observed between the ω_n obtained using SVIM and the exact values obtained by previous researchers.

Keywords: single variable shear deformation beam theory, Stodola-Vianello iteration method, natural vibration frequency forced vibration, vibration mode.

1. Introduction

Beams are structural members that are extensively used in aerospace, mechanical, civil and automobile engineering. They are known to have longitudinal dimensions that are much larger than the cross-sectional dimensions. They carry transverse loadings by the development of flexural deformations. They can be subjected to static, dynamic and/or in-plane loads and hence can develop flexural responses. The focus of this paper is on dynamic response of beams.

Beams that are subjected to time varying forces are prone to resonance failures when the inherent natural frequencies of the beam are equal to the frequencies of external excitation due to the applied load. The natural frequencies of structures and beams have been found to depend upon their stiffness and intertial properties.

In order to avoid resonant failures of beams, it is a vital step to conduct natural frequency analysis in order to determine the resonant frequencies at which failures would occur upon coincidence with the dynamic loading frequencies.

The dynamic behaviour of beam is dependent on the geometry of the beam among other

factors. When the ratio of depth of the cross-section to the span l of the beam is less than or equal to 0.05, the beam is called a thin or slender beam. When the ratio of depth to span is greater than 0.05, the beam becomes moderately thick or thick. Beams are thus classified as thin when $h/l \leq 0.05$, thick if $h/l > 0.05$.

Several theories have been developed for beam vibration analysis. The Euler-Bernoulli beam theory (EBBT) is the classical beam theory (CBT) which has been found suitable for thin beams. The theory has been formulated for static flexural, dynamic and buckling problems [1-5]. The EBBT was derived using the Euler-Bernoulli-Navier hypothesis that vertical lines on the plane cross-sections that are initially normal to the longitudinal neutral axis before deformation would remain plane and normal to the longitudinal neutral axis after deformation due to the applied loads [2], [4]. This Navier-Euler-Bernoulli orthogonality hypothesis implies that there are no transverse shear deformations. Shear strains which can produce non-planar cross-sectional deformations are ignored in the EBBT [6].

The formulation of EBBT equations thus discountenance transverse shear deformations rendering the theory unsuitable for moderately thick and thick beams where transverse shear deformation effect could significantly affect the vibration characteristics.

Erdelyi and Hashemi [7] used the EBBT for deriving dynamic stiffness matrix (DSM) to relate sinusoidally varying loads to sinusoidally varying displacements at the beam ends for delaminated multilayered beams. Their DSM was then used to study the natural frequencies and mode shape of two-layer beam systems.

In order to provide other beam theories to incorporate shear deformation effects, Timoshenko [8] derived the first order shear deformation beam theory called the Timoshenko beam theory (TBT) which has been extended by several scholars to describe vibration and buckling beam behaviours.

TBT considers the deformations to be made of the sum of bending and shear deformations and thus is applicable to moderately thick beams. However, the TBT yields constant transverse shear strain distribution across the beam depth, thus resulting in a non-zero shear stress at the beam's top and bottom surfaces in violation of the transverse shear stress-free boundary conditions at the beam's top and bottom surfaces (Ike, [2]) shear modification factors which are problem dependent and dependent on cross-sectional shape were introduced in order to ensure that the accurate strain energy of deformation is obtained in the theory.

Cowper [9] researched on the improvements of the TBT and developed analytical expressions for the shear stress modification factors in terms of the cross-section geometry and Poisson's ratio. Mindlin improved the TBT by simplifying the transverse shear strain as a constant distribution through the beam depth profile [6]. Mindlin used a shear coefficient, k_s, to properly represent the strain energy of the beam and to include the effect of non-constant shear stresses and strains. Values of k_s for different cross-sectional shapes were presented by Cowper [9].

Several researches were inspired by the need to solve the issues associated with the EBBT, TBT and their improvements. These researches led to the development of shear deformation theories (SDTs), higher order shear deformation theories (HoSDTs) and refined beam theories (RBTs).

SDTs, HoSDTs and RBTs were all developed by several investigators with the principal concern to satisfy the transverse shear stress free conditions at the top and bottom beam surfaces ($z = \pm h/2$) while also incorporating the transverse shear deformation into the formulation. Shear shape functions were used in various formulations of SDTs, HoSDTs and RBTs in order to ensure that the shear stress free boundary conditions are satisfied. These shear shape functions used are derived such that the shear stress free boundary conditions are satisfied. The commonly used functions are: polynomials, trigonometric, hyperbolic and exponential functions.

Apart from propositions on the several shear shape functions researchers have constructed refined beam theories that seek to reduce the number of unknown displacement and/or stress terms in beam theories. Touratier [10] presented exact solutions to the thick beam vibration problem.

Shear deformable beam theories have been studied by Pakhare et al. [11]; Shimpi et al. [12]; Vascocelos et al. [13], Levinson, [14]; Murty [15] and Shi and Voyiadjis [16]. Levinson [14] presented a third order shear deformable rectangular beam theory for bending analysis but did not consider vibration behaviour. Murty [15] derived a shear deformation theory for the dynamic analysis of beams and used it to develop natural frequencies of vibrating beams under simple supports. Shi and Voyiadjis [16] derived a sixth-order theory of shear deformable beams using the methods of variational calculus. Stephen and Levinson [17] derived a shear deformable theory to incorporate shear strain effects.

Ghugal and Sharma [18] developed a shear deformable beam theory for bending and vibration analysis using hyperbolic shear function to ensure the satisfaction of the transverse shear stress-free boundary conditions. Pakhare et al. [11] used the methods of variational calculus to develop variationally consistent and inconsistent equations for the natural transverse vibration characteristics of shear deformable beams made of homogeneous isotropic linear elastic materials. They used Fourier series method to obtain accurate eigensolutions for the natural frequencies.

Shimpi et al. [12] developed shear deformation beam theory using a single displacement variable as the unknown in their displacement based formulation. Thai et al. [19] presented a deformation theory for non-local beams.

Higher order shear deformable beam theories have been investigated by Razouki et al. [20], Zioù et al. [21] and Heyliger and Reddy [22]. Heyliger and Reddy [22] developed finite element stiffness and inertial matrices based on a higher order shear deformation beam theory for the bending and vibration problems. Zioù et al. [21] used a higher order shear deformable theory for the static bending analysis of functionally graded (FG) beams, but did not extend their work to vibration analysis.

Razouki et al. [20] also used the differential transform method (DTM) to solve bending problems of thick FG beams formulated using HoSDT, but they did not consider vibration problems. Ghugal and Shimpi [23] presented a concise review of refined shear deformation theories for isotropic and anisotropic laminated beams. Sayyad and Ghugal [24] developed single variable refined beam theories for the bending, buckling and natural vibration analysis of homogeneous beams.

Ibearugbulem et al. [25] used the principle of energy minimization to determine the natural vibrating frequencies of refined beams. Shimpi et al. [26] developed simple two variable refined beam theory for isotropic, homogeneous, rectangular beams, but did not consider vibration cases. Sayyad [27] presented a comparison of different RBTs for the bending and natural vibration behaviour of thick beams.

This study adopts the Stodola-Vianello vibration method for solving the resulting thick beam vibrating problem. The Stodola-Vianello iteration method has been sparsely used.

Dong et al. [28] presented applications of the Stodolla-Vianello and Gram-Schmidt schemes for solving eigen problems of frequencies and mode shapes. They found reduction in computational effort and computer memory space as advantages of the iterative method.

Chennit et al. [29] used the Stodola-Vianello iteration method to determine the periods and mode shapes for uniform shear wall buildings, which are important dynamic characteristics essential for the analysis of seismic response of multi-storey buildings. Their research provided new database of vibration modes of multi-storey buildings with reinforced concrete (RC) shear walls.

Ike [30], [31] used the Stodola-Vianello iteration method to obtain exact buckling load solutions for simply supported Euler-Bernoulli beam on Winkler and Pasternak foundations respectively. Sinusoidal buckling shape functions were used in the derived Stodolla-Vianello iteration equations to obtain the next eigenfunction and the convergence rule at the $(n + 1)$th iteration was used to find the eigenvalues which were used to obtain the exact buckling loads at any nth buckling mode. Ofondu et al. [32] applied the SVIM for the critical buckling load determination of Euler columns.

In another study Ike [33] used the Stodola-Vianello iteration method to determine approximate

buckling load solutions using polynomial buckling shape functions in the Stodola-Vianello iteration formula for simply supported thin beam on Winkler foundation. Ike et al. [34], [35] have also applied the Stodolla-Vianello iteration method to obtain sufficiently accurate approximate buckling solutions for clamped-clamped thin beam on Winkler and Pasternak foundations respectively. In a recent work Ike [36] used the SVIM for the critical buckling load solutions of Euler-Bernoulli beams on two-parameter foundations using polynomial basis functions.

A review of literature reveals that very few studies have been done on the subject of transverse vibrations of thick beams modelled using shear deformable theories; and the Stodola-Vianello method has never been used for solving the governing differential equations of vibration. The aim of this paper is to derive from first principles Shimpi's single variable shear deformation theory for transverse flexural vibrations for free and forced vibration, and then apply Stodolla-Vianello iteration method for the first time, to solve the resulting governing equation of motion for simply supported boundary conditions and natural harmonic vibrations.

2. Novelty of the study

The novelty of the study are:
– The first principles systematic presentation of the derivation of the governing equation of motion using the method of differential equation of dynamic equilibrium of an infinitesimal beam element and consideration of strain-displacement equation, stress-strain equations of linear elasticity and Shimpi et al. [12] assumptions of the displacement field components.
– The Stodola-Vianello iteration method is applied for the first time in a systematic way to formulate the Stodola-Vianello iteration equations and to obtain convergent solutions for simply supported boundary conditions.

2.1. Advantages and disadvantages of the Stodola-Vianello iteration method (SVIM)

The SVIM is adopted in this study as the solution method because of the previously demonstrated effectiveness and its merits which include:
– The SVIM simplifies the BVP of solving the governing equation of motion subject to boundary conditions to an algebraic iterative problem which usually is simpler to solve, and hence demands less computational rigour.
– The SVIM results in an iterative problem that involves the evaluation of successive integration problems.
– The method gives exact eigenvalue solutions which can be used to find the nth eigenvalue when the exact eigenfunction is used.
– The method can be used to obtain approximate solutions with the use of approximate expressions for the shape function, and use of high number of iterations.
However, the SVIM has the following limitations of
– The tediousness of the calculation increases as the number of iterations increases.
– Extreme difficulty involved in handling anisotropic and heterogeneous beam materials.

2.2. Theoretical framework of the Shimpi's single variable shear deformation beam theory

Shimpi et al. [12] assumed the displacement field components in the x, y and z coordinate directions (u, v and w) as:

$$u = -z \frac{dw_b}{dx} + \left(\frac{z}{4} - \frac{5}{3} \frac{z^3}{h^2} \right) \frac{dw_s}{dx}, \tag{1a}$$
$$v = 0, \tag{1b}$$
$$w = w_b + w_s, \tag{1c}$$

where w_b is the bending displacement of a point on the midplane of the beam, w_s is the shear displacement of a point on the midplane of the beam, h is the thickness of the beam.

The formulation neglects the in-plane displacement since the bending and stretching deformations are decoupled for the cases of homogeneous, isotropic beams. Also, a quadratic distribution of transverse shear stress across the beam depth such that transverse shear stress free boundary conditions are satisfied is obtained using the displacement field in Eqs. (1a) and (1c).

The normal stresses and shear stresses are expressed using the material constitutive laws for homogeneous isotropic behaviour as:

$$
\begin{aligned}
\sigma_{xx} &= E\varepsilon_{xx}, \\
\tau_{xz} &= G\gamma_{xz}, \\
\sigma_{yy} &= 0, \\
\sigma_{xz} &= 0, \\
\tau_{xy} &= 0, \\
\tau_{yz} &= 0,
\end{aligned}
\tag{2}
$$

where E is the Young's modulus of elasticity, G is the shear modulus or modulus of rigidity. σ_{xx}, σ_{yy}, σ_{zz} are normal stresses in the x, y and z directions. ε_{xx}, ε_{yy}, ε_{zz}, are normal strains in the x, y and z directions. γ_{xy}, γ_{yz} and γ_{xz} are shear strains in the xy, yz and xz planes. τ_{xz}, τ_{yz} and τ_{xz} are shear stresses on the xy, yz and xz planes.

From the strain-displacement relations of linear small displacement elasticity theory, the normal and shear strains are expressed in terms of the displacement components as:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = -z\frac{d^2 w_b}{dx^2} + \left(\frac{z}{4} - \frac{5z^3}{3h^2}\right)\frac{d^2 w_s}{dx^2}, \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \left(\frac{5}{4} - \frac{5z^2}{h^2}\right)\frac{dw_s}{dx}, \\
\gamma_{xy} &= 0, \\
\gamma_{yz} &= 0, \\
\varepsilon_{yy} &= 0, \\
\varepsilon_{zz} &= 0.
\end{aligned}
\tag{3}
$$

The differential equations of dynamic equilibrium of two-dimensional (2D) problems of elasticity when body forces are disregarded are:

$$
\frac{d\sigma_{xx}}{dx} + \frac{d\tau_{xz}}{dz} = \rho\ddot{u} = \rho\frac{d^2 u}{dt^2},
\tag{4a}
$$

$$
\frac{d\tau_{xz}}{dx} + \frac{d\sigma_{zz}}{dz} = \rho\ddot{w} = \rho\frac{d^2 w}{dt^2},
\tag{4b}
$$

where the dots over u and w denote time derivatives, and ρ is the mass density

The differential equations of dynamic equilibrium are expressed in terms of stress resultants by multiplying Eq. (4a) with z and then integrating the resulting product expression over the beam cross-section and also by integrating Eq. (4b) over the cross-section and applying the boundary conditions given in Eqs. (5) and (6):

$$
\tau_{xz} = 0, \qquad z = \pm h/2,
\tag{5}
$$

$$
\sigma_{zz} = -q(x), \quad z = h/2.
\tag{6}
$$

Thus, the dynamic equilibrium equations are given by Eq. (7) and (8):

$$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} z \left\{ \left(\frac{d\sigma_{xx}}{dx} + \frac{d\tau_{xz}}{dz} \right) - \rho \ddot{u} \right\} dy dz = 0, \tag{7}$$

$$\iint_{A} \left\{ \frac{d\tau_{xz}}{dx} + \frac{d\sigma_{zz}}{dz} - \rho \ddot{w} \right\} dA = 0, \tag{8}$$

where A is the cross-sectional area of the beam:

$$-\frac{b}{2} \le y \le \frac{b}{2}, \quad -\frac{h}{2} \le z \le \frac{h}{2}, \quad dA = dy\, dz. \tag{9}$$

Thus, using the linearity properties of integrations, expanding Eqs. (7) and (8) give:

$$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} z \frac{d\sigma_{xx}}{dx}\, dy dz + \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} z \frac{d\tau_{xz}}{dz}\, dy dz - \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \rho \frac{\partial^2 u}{\partial t^2}\, dy dz = 0, \tag{10}$$

$$\int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \frac{d\tau_{xz}}{dx}\, dy dz + \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \frac{d\sigma_{zz}}{dz}\, dy dz - \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \rho \frac{d^2 w}{dt^2}\, dy dz = 0. \tag{11}$$

Simplifying Eq. (11) gives:

$$\int_{-h/2}^{h/2} \frac{d\tau_{xz}}{dx} b dz + \int_{-h/2}^{h/2} b \frac{d\sigma_{zz}}{dz} dz - \rho \frac{d^2 w}{dt^2} \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} dy dz = 0. \tag{12}$$

Further simplification of Eq. (12) gives:

$$\frac{d}{dx} \int_{-h/2}^{h/2} b\tau_{xz}\, dz + \int_{-h/2}^{h/2} \frac{bd\sigma_{zz}}{dz}\, dz - \rho A \frac{d^2 w}{dt^2} = 0. \tag{13}$$

Simplifying Eq. (13) further gives:

$$\frac{dV}{dx} + q(x) - \rho A \ddot{w} = 0. \tag{14}$$

Using Eq. (1c) in Eq. (14) gives:

$$\frac{dV}{dx} + q(x) - \rho A (\ddot{w}_s + \ddot{w}_b) = 0, \tag{15}$$

where:

$$V = \int_{-h/2}^{h/2} \tau_{xz} b dz = \int_{-h/2}^{h/2} G\gamma_{xz} b dz. \tag{16}$$

V is the shear force distribution:

$$V = Gb \int_{-h/2}^{h/2} \gamma_{xz} \, dz. \tag{17}$$

Hence, substituting the expression for γ_{xz} in Eq. (17) gives:

$$V = Gb \int_{-h/2}^{h/2} \left(\frac{5}{4} - \frac{5z^2}{h^2} \right) \frac{dw_s}{dx} \, dz. \tag{18}$$

Simplifying Eq. (18) gives:

$$V = Gb \frac{dw_s}{dx} \int_{-h/2}^{h/2} \left(\frac{5}{4} - \frac{5}{h^2} z^2 \right) dz. \tag{19}$$

Integrating Eq. (19) gives:

$$V = Gb \frac{dw_s}{dx} \left[\frac{5z}{4} - \frac{5}{h^2} \frac{z^3}{3} \right]_{-h/2}^{h/2}. \tag{20}$$

Evaluation of Eq. (20) gives:

$$V = Gb \frac{dw_s}{dx} \left(\frac{5}{4} \left(\frac{h}{2} \right) - \frac{5}{3h^2} \left(\frac{h}{2} \right)^3 \right) \times 2. \tag{21}$$

Or:

$$V = Gb \frac{dw_s}{dx} \left(\frac{5}{4} \frac{h}{2} - \frac{5}{3h^2} \frac{h^3}{8} \right) \times 2. \tag{22}$$

Thus, further simplification of Eq. (22) gives:

$$V = Gb \frac{dw_s h}{dx} \left(\frac{5}{8} - \frac{5}{24} \right) \times 2. \tag{23}$$

Simplifying, Eq. (23) gives:

$$V = Gbh \frac{dw_s}{dx} \left(\frac{20}{24} \right), \tag{24}$$

hence:

$$V = \frac{5}{6} Ghb \frac{dw_s}{dx}, \tag{25a}$$

or:

$$V = \frac{5}{6} GA \frac{dw_s}{dx}. \tag{25b}$$

Similarly, Eq. (10) gives:

$$\frac{d}{dx} \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} z\, \sigma_{xx}\, dydz + \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} Gz \frac{d}{dz}\left(\frac{5}{4} - \frac{5z^2}{h^2}\right) dydz$$
$$-\rho \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \frac{d^2}{dt^2}\left(-z\frac{dw_b}{dx} + \left(\frac{z}{4} - \frac{5}{3}\frac{z^3}{h^2}\right)\frac{dw_s}{dx}\right) dydz = 0. \tag{26}$$

Hence:

$$\frac{dM}{dx} - V - \rho I \frac{d}{dx}(\ddot{w}_b) = 0, \tag{27}$$

where:

$$M(x) = \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \sigma_{xx}\, zdydz\,, \tag{28}$$

$$M(x) = \int_{-h/2}^{h/2} \sigma_{xx}\, zbdz\,. \tag{29}$$

M is the bending moment distribution:

$$M(x) = -EI\frac{d^2 w_b}{dx^2}, \tag{30}$$

$$I = \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} z^2\, dz\, dy\,, \tag{31}$$

$$I = \int_{-h/2}^{h/2} b\, z^2\, dz\; = \frac{bh^3}{12}, \tag{32}$$

$$A = \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} dy\, dz = bh\,. \tag{33}$$

Substituting Eqs. (25b) and (30) into Eqs. (15) and (27) give as follows:

$$w_s = \frac{5}{6GA}\left(-EI\frac{d^2 w_b}{dx^2} + I\rho\ddot{w}_b\right). \tag{34}$$

Hence the displacement field equations are expressed in terms of only one unknown displacement function $w_b(x,t)$ since Eq. (34) enables the expression of the shear displacement $w_s(x,t)$ in terms of $w_b(x,t)$.

Then:

$$u(x,t) = -z\frac{dw_b}{dx} + \frac{6}{5GA}\left(\frac{z}{4} - \frac{5z^3}{3h^2}\right)\left(-EI\frac{d^3 w_b}{dx^3} + I\rho\frac{d\ddot{w}_b}{dx}\right), \tag{35}$$

$$w(x,t) = w_b + \frac{6}{5GA}\left(-EI\frac{d^2 w_b}{dx^2} + I\rho\ddot{w}_b\right). \tag{36}$$

Hence, from Eqs. (15), (27) and (34) it is obtained after algebraic simplifications and re-arrangements, the governing equation of dynamic equilibrium of a SVSDBT as follows:

$$EI\frac{d^4 w_b}{dx^4} + \rho A\ddot{w}_b + \frac{6\rho^2 I}{5G}\ddot{w}_b - \rho I\left(1 + \frac{6E}{5G}\right)\frac{d^2\ddot{w}_b}{dx^2} = q(x,t). \tag{37}$$

2.3. Governing equation

Eq. (37) can be re-expressed in the expanded form as Eq. (38):

$$EI\frac{\partial^4 w_b}{\partial x^4} - \rho I\left(1 + \frac{12(1+\mu)}{5}\right)\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_b}{\partial x^2}\right) + \rho A\frac{\partial^2 w_b}{\partial t^2} + \frac{\rho^2 I}{E}\left(\frac{12(1+\mu)}{5}\right)\frac{\partial^4 w_b}{\partial t^4} \tag{38a}$$
$$= q(x,t).$$

For free vibrations, the PDE is the homogeneous form of Eq. (38a) given by Eq. (38b):

$$EI\frac{\partial^4 w_b}{\partial x^4} - \rho I\left(1 + \frac{12(1+\mu)}{5}\right)\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_b}{\partial x^2}\right) + \rho A\frac{\partial^2 w_b}{\partial t^2} + \frac{\rho^2 I}{E}\left(\frac{12(1+\mu)}{E}\right)\frac{\partial^4 w_b}{\partial t^4} = 0. \tag{38b}$$

The boundary conditions at the simply supported ends $x = 0$, $x = l$ are given by Eqs. (39a) and (39b):

$$\left(w_b - h^2\frac{(1+\mu)}{5}\frac{d^2 w_b}{\partial x^2}\right)\Bigg|_{x=0} = 0,$$
$$\left(w_b - \frac{h^2(1+\mu)}{5}\frac{d^2 w_b}{\partial x^2}\right)\Bigg|_{x=l} = 0. \tag{39}$$

For harmonic vibrations, let $w_b(x,t)$ be expressible as the infinite series:

$$w_b(x,t) = \sum_{n=1}^{\infty} F_n(x)\sin\omega_n t, \tag{40}$$

where $F_n(x)$ is the modal function. By differentiation:

$$\frac{\partial^2 w_b}{\partial x^2} = \sum_{n=1}^{\infty} F''_n(x)\sin\omega_n t, \tag{41}$$

$$\frac{\partial^4 w_b}{\partial x^4} = \sum_{n=1}^{\infty} F_n^{iv}(x)\sin\omega_n t, \tag{42}$$

$$\frac{\partial^2 w_b}{\partial t^2} = \sum_{n=1}^{\infty} -\omega_n^2 F_n(x)\sin\omega_n t, \tag{43}$$

$$\frac{\partial^2}{\partial t^2}\frac{\partial^2 w_b}{\partial x^2} = \sum_{n=1}^{\infty} -\omega_n^2 F''_n(x)\sin\omega_n t, \tag{44}$$

$$\frac{\partial^4 w_b}{\partial t^4} = \sum_{n=1}^{\infty} \omega_n^4 F_n(x) \sin\omega_n t. \tag{45}$$

Hence, Eq. (38) becomes:

$$EI \sum_{n=1}^{\infty} F_n^{iv} \sin\omega_n t - \rho I \left(1 + \frac{12(1+\mu)}{5}\right) \sum_{n=1}^{\infty} -\omega_n^2 F''_n(x) \sin\omega_n t$$
$$+ \rho A \sum_{N=1}^{\infty} -\omega_n^2 F_n(x) \sin\omega_n t + \frac{\rho^2 I}{E}\left(\frac{12(1+\mu)}{5}\right) \sum_{n=1}^{\infty} \omega_n^4 F_n(x) \sin\omega_n t = 0. \tag{46}$$

Simplifying Eq. (46) gives:

$$\sum_{n=1}^{\infty} \left\{ EI \, F_n^{iv}(x) + \rho I \omega_n^2 \left(1 + \frac{12(1+\mu)}{5}\right) F''_n(x) - \rho A \omega_n^2 F_n(x) \right. $$
$$\left. + \frac{\rho^2 I}{E} \omega_n^4 F_n \frac{12(1+\mu)}{5} \right\} \sin\omega_n t = 0. \tag{47}$$

The modal equation is the ordinary differential equation in $F_n(x)$ given by Eq. (48):

$$EI \, F_n^{iv}(x) + \rho I \omega_n^2 \left(1 + \frac{12(1+\mu)}{5}\right) F''_n(x) + \left(\frac{12(1+\mu)}{5}\rho^2 I \omega_n^4 - \rho A \omega_n^2\right) F_n(x) = 0. \tag{48}$$

2.4. Stodola-Vianallo iteration method

Rearranging Eq. (48) gives:

$$F_n^{iv}(x) = -\frac{\rho I \omega_n^2}{EI}\left(1 + \frac{12(1+\mu)}{5}\right) F''_n(x) - \left(\frac{12(1+\mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right) F_n(x). \tag{49}$$

By successive integrations:

$$F'''_n(x) = -\frac{\rho I \omega_n^2}{EI}\left(1 + \frac{12(1+\mu)}{5}\right) F'_n(x)$$
$$- \left(\frac{12(1+\mu)}{5}\frac{\rho^2 \omega_n^4 I}{EI} - \frac{\rho A \omega_n^2}{EI}\right) \int_0^x F_n(x) \, dx + c_1, \tag{50}$$

where c_1 is an integration constant.

$$F''_n(x) = \frac{-\rho I \omega_n^2}{EI}\left(1 + \frac{12(1+\mu)}{5}\right) F_n(x)$$
$$- \left(\frac{12(1+\mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right) \int_0^x \int_0^x F_n(x) \, dx dx + c_1 x + c_2, \tag{51a}$$

where c_2 is the second integration constant.

$$F''_{n+1}(x) = \frac{-\rho I \omega_n^2}{EI}\left(1 + \frac{12(1 + \mu)}{5}\right)F_n(x)$$
$$-\left(\frac{12(1 + \mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right)\int_0^x\int_0^x F_n(x)\,dxdx + c_1 x + c_2. \tag{51b}$$

Integrating Eq. (51a) gives Eq. (52):

$$F'_n(x) = \frac{-\rho I \omega_n^2}{EI}\left(1 + \frac{12(1 + \mu)}{5}\right)\int_0^x F_n(x)dx$$
$$-\left(\frac{12(1 + \mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right)\int_0^x\int_0^x\int_0^x F_n(x)\,dxdxdx + \frac{c_1 x^2}{2} + c_2 x + c_3, \tag{52}$$

where c_3 is the third constant of integration.
Integrating Eq. (52) gives Eq. (53):

$$F_n(x) = \frac{-\rho I \omega_n^2}{EI}\left(1 + \frac{12(1 + \mu)}{5}\right)\int_0^x\int_0^x F_n(x)dxdx$$
$$-\left(\frac{12(1 + \mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right)\int_0^x\int_0^x\int_0^x\int_0^x F_n(x)\,dxdxdxdx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x$$
$$+ c_4, \tag{53}$$

where, c_4 is the fourth integration constant.
The Stodola-Vianello iteration equation then becomes:

$$F_{n+1}(x) = \frac{-\rho I \omega_n^2}{EI}\left(1 + \frac{12(1 + \mu)}{5}\right)\int_0^x\int_0^x F_n(x)\,dxdx$$
$$-\left(\frac{12(1 + \mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right)\int_0^x\int_0^x\int_0^x\int_0^x F_n(x)\,dxdxdxdx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x$$
$$+ c_4. \tag{54}$$

3. Results

$F_n(x)$ for the nth vibration mode which satisfies the simply supported boundary conditions is:

$$F_n(x) = \bar{F}_n \sin\frac{n\pi x}{l}. \tag{55}$$

Since:

$$F_n(0) = 0 = F_n(l) = 0,$$
$$F''_n(0) = 0 = F''_n(l) = 0, \tag{56}$$

where $\bar{F}_n$ is the amplitude of $F_n(x)$ for the nth vibration mode:

$$F_{n+1}(x) = \frac{-\rho I \omega_n^2}{EI}\left(1 + \frac{12(1+\mu)}{5}\right)\bar{F}_n \int_0^x \int_0^x \sin\frac{n\pi x}{l}\, dx dx$$

$$-\bar{F}_n\left(\frac{12(1+\mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right)\int_0^x\int_0^x\int_0^x\int_0^x \sin\frac{n\pi x}{l}\, dx dx dx dx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x$$

$$+ c_4,$$ (57a)

$$F_{n+1}(x) = \bar{F}_n \frac{\rho I \omega_n^2}{EI}\left(1 + \frac{12(1+\mu)}{5}\right)\frac{l^2}{(n\pi)^2}\sin\frac{n\pi x}{l}$$

$$-\bar{F}_n\left(\frac{12(1+\mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right)\left(\frac{l}{n\pi}\right)^4\sin\frac{n\pi x}{l} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4.$$ (57b)

Applying the boundary conditions:

$$F_{n+1}(x=0) = 0 = c_4,$$
$$F''_{n+1}(x=0) = 0 = c_2,$$
$$F''_{n+1}(x=l) = 0 = c_1 l = 0,$$
$$F_{n+1}(x=l) = 0 = c_3 l.$$ (58)

Hence:

$$F_{n+1}(x) = \frac{\rho I \omega_n^2}{EI}\bar{F}_n\left(1 + \frac{12(1+\mu)}{5}\right)\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}$$

$$-\bar{F}_n\left(\frac{12(1+\mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right)\left(\frac{l}{n\pi}\right)^4\sin\frac{n\pi x}{l}.$$ (59)

At convergence:

$$F_{n+1}(x) = F_n(x).$$ (60)

So:

$$\bar{F}_n\sin\frac{n\pi x}{l} = \frac{\rho I \omega_n^2}{EI}\bar{F}_n\left(1 + \frac{12(1+\mu)}{5}\right)\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}$$

$$-\bar{F}_n\left(\frac{12(1+\mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right)\left(\frac{l}{n\pi}\right)^4\sin\frac{n\pi x}{l}.$$ (61)

Division by $\bar{F}\sin\frac{n\pi x}{l}$ gives:

$$1 = \frac{\rho I \omega_n^2}{EI}\left(1 + \frac{12(1+\mu)}{5}\right)\left(\frac{l}{n\pi}\right)^2 - \left(\frac{12(1+\mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right)\left(\frac{l}{n\pi}\right)^4.$$ (62)

Rearranging the equation gives:

$$\left(\frac{n\pi}{l}\right)^4 = \frac{\rho I \omega_n^2}{EI}\left(1 + \frac{12(1+\mu)}{5}\right)\left(\frac{n\pi}{l}\right)^2 - \left(\frac{12(1+\mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \frac{\rho A \omega_n^2}{EI}\right).$$ (63)

Further re-arrangement gives:

$$\frac{12(1+\mu)}{5}\frac{\rho^2 I \omega_n^4}{EI} - \left(\frac{\rho A}{EI} + \frac{\rho I}{EI}\left(1 + \frac{12(1+\mu)}{5}\right)\left(\frac{n\pi}{l}\right)^2\right)\omega_n^2 + \left(\frac{n\pi}{l}\right)^4 = 0. \tag{64}$$

Lets define the dimensionless frequency parameter θ_n as:

$$\theta_n = \omega_n l^2 \sqrt{\frac{\rho A}{EI}}. \tag{65}$$

Then the characteristic equation is expressed in terms of θ_n as:

$$\frac{12(1+\mu)}{5}\frac{I^2 \theta_n^4}{A^2 l^4} - \left(\frac{I}{A}\left(1 + \frac{12(1+\mu)}{5}\right)\left(\frac{n\pi}{l}\right)^2 + 1\right)\theta_n^2 + (n\pi)^4 = 0. \tag{66}$$

But for rectangular cross-sections:

$$I = \frac{bh^3}{12}, \tag{67}$$

$$A = bh, \tag{68}$$

$$\frac{I}{A} = \frac{h^2}{12}. \tag{69}$$

So:

$$\left(\frac{1+\mu}{60}\right)\left(\frac{h}{l}\right)^4 \theta_n^4 - \left(\frac{h^2}{l^2}\left(\frac{1}{12} + \frac{1+\mu}{5}\right)(n\pi)^2 + 1\right)\theta_n^2 + (n\pi)^4 = 0. \tag{70}$$

This can be considered a quadratic equation in θ_n^2 and the roots of the equation can be found using the quadratic formula. Using θ_n^2, ω_n^2 can be found as:

$$\theta_n^2 = \omega_n^2 l^4 \left(\frac{\rho A}{EI}\right), \tag{71a}$$

$$\omega_n^2 = \theta_n^2 \frac{EI}{\rho A l^4}. \tag{71b}$$

So:

$$\omega_n = \theta_n \sqrt{\frac{EI}{\rho A l^4}}. \tag{72}$$

The results for θ_n are presented in terms of various values of l/h and for $\mu = 0.30$ in Tables 1, 2, 3 and 4. The variations of θ_n with l/h are also presented for the first three mode of vibration in Figs. 1, 2, and 3.

4. Discussion

In this paper, Shimpi's single variable shear deformation beam theory for forced and free transverse flexural vibrations have been derived from first principles and then the Stodola-Vianello iteration method applied for the first time, in a novel systematic way to solve the resulting governing equation of motion for simply supported boundary conditions and natural harmonic vibrations. The derivation was based on the differential equation of dynamic equilibrium

method, rather than the commonly found variational calculus procedure. Differential equations of dynamic equilibrium, stress-strain relations and the strain displacement relations of small displacement linear elasticity were simultaneously applied with the Shimpi et al. [12] expressions for the assumed displacement field in order to determine the partial differential equations of motion. The obtained boundary value problem (BVP) which is a non-homogeneous PDE for force vibration, simplifies to a homogeneous PDE for free vibrations due to the absence of the excitation forcing force in the case of free (natural) vibration. The study focused on free vibration, and hence entailed the solution of the homogeneous PDE using Stodola-Vianello iterative method. For the simply supported thick beam vibration problem, the BVP is represented by Eq. (38b) and the boundary conditions Eqs. (39a) and (39b). For harmonic vibrations and expected harmonic response the PDE simplifies to a fourth order ordinary differential equation (ODE) as the modal equation given by Eq. (48). This fourth order modal equation is expressed using SVIM as the system of iterative equations – Eqs. (51b) and (54).

The use of sinusoidal eigenfunctions at the nth vibration mode which is the exact eigenfunction for a simply supported beam in the Stodola-Vianello iteration equation led to the determination of the four constants of integration. The application of the convergence rule at the nth vibration mode as expressed by Eq. (61) resulted in the algebraic eigenvalue problem given by Eq. (63). Further rearrangements of Eq. (63) and simplification for beams of rectangular cross-sections gave the fourth degree characteristic eigenvalue equation in terms of θ_n as Eq. (71). Hence ω_n is found expressed in terms of θ_n as Eq. (72c). The results for θ_n are presented in terms of various values of l/h for $\mu = 0.30$ in Tables 1, 2, 3 and 4 and in Figs. 1, 2, and 3.

Tables 1-4 and Figs. 1-3 also compare the obtained values of θ_n for various vibration modes $n = 1$, $n = 2$, $n = 3$ and $n = 4$ with previous values presented by Shimpi et al. [26], Touratier [10], Timoshenko [8] and Pakhare et al. [11].

Table 1. Dimensionless free vibration frequencies θ_n of shear deformable thick simply supported beams for $\mu = 0.30$ and various values of l/h for $n = 1$

Reference	l/h				
	100	20	10	20/3	5
Present study (SVIM)	9.8679 (0.00 %)	9.8281 (0.00 %)	9.7075 (0.00 %)	9.5180 (–0.01 %)	9.2740 (–0.01 %)
Pakhare et al. [11] Shimpi et al. [26] Navier's method (SVSDBT)	9.8679 (0.00 %)	9.8281 (0.00 %)	9.7075 (0.00 %)	9.5180 (–0.00 %)	9.2740 (–0.00 %)
Touratier [10]	9.8679	9.8282	9.7077	9.5186	9.2752
EBBT (Timoshenko [8])	9.8692 (0.01 %)	9.8595 (0.32 %)	9.8293 (1.25 %)	9.7795 (2.74 %)	9.7112 (4.70 %)
Pakhare, et al. [11] Shimpi et al. [26] Navier's method TVSDBT	9.8679 (0.00 %)	9.8281 (0.00 %)	9.7015 (0.00 %)	9.5182 (0.00 %)	9.2745 (-0.01 %)

Table 1 shows that the present study is identical with previous studies by Shimpi et al. [26] and Pakhare et al. [11] who both used Navier's series method to solve the single variable shear deformable beam problem. Table 1 also illustrates that the present SVIM results are identical with the Navier solutions of the flexural vibration results of thick beams modelled using two-variable shear deformable beam theory as presented by Pakhare et al. [11] and Shimpi et al. [26]. Table 1 further compares the SVIM present results with exact results presented by Touratier [10] and shows that for l/h equal to 10, 20, and 100, the present SVIM results are identical with exact results by Touratier [10], but –0.01 % different from the exact results for l/h equal to 5 and 20/3. The Table 1 further illustrates that the EBBT results differ from the exact results of Touratier [10] by 0.01 % for l/h equal to 100; 0.32 % for l/h equal to 20 and the errors increase to 4.70 % for
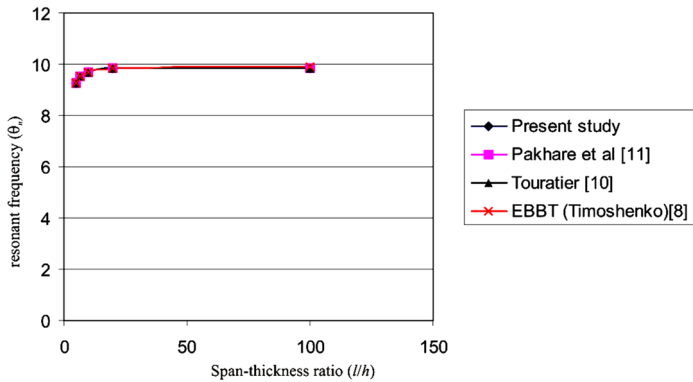
l/h equal to 5.0.

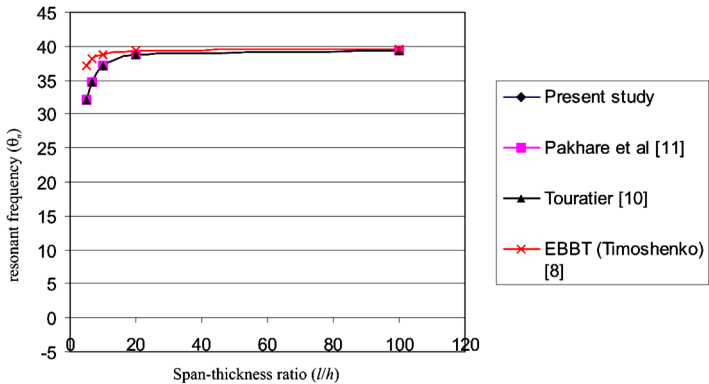**Fig. 1.** Graph of θ_n vs l/h for $n = 1$ (simply supported thick beams)

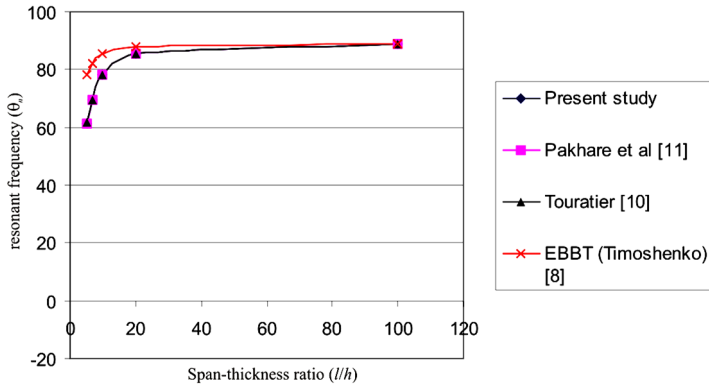**Fig. 2.** Graph of θ_n vs l/h for $n = 2$ (simply supported thick beams)

Fig. 3. Graph of θ_n vs l/h for $n = 3$ (simply supported thick beams)

Similar observations are noted for Table 2 which shows the natural frequency results for the second mode of flexural vibration for the thick beam problem. Table 2 shows that the SVIM results for θ_n are identical with the SVSDBT results obtained using Navier's method by Pakhare et al. [11] and Shimpi et al. [26] and in close agreement with TVSDBT solutions using Navier method by Pakhare et al. [11] and Shimpi et al. [26]. Comparison of present results with Touratier [10] exact solution shows a negligible difference of 0 % for $l/h = 100$, 0 % for $l/h = 20$, –0.01 % for $l/h = 10$, –0.04 % for $l/h = 20/3$ and –0.09 % for $l/h = 5$. The EBBT results of Timoshenko [8]

presented in Table 2 show the unsuitability of the EBBT method for eigensolution of the problem especially for $l/h < 10$ due to solution errors exceeding 4.70 %.

Table 2. Dimensionless free (natural) vibration frequencies θ_n of shear deformable thick simply supported beams for $\mu = 0.30$ and various values of l/h for $n = 2$

Reference	l/h				
	100	20	10	20/3	5
Present study (SVIM)	39.4517 (0.00 %)	38.8299 (0.00 %)	37.0962 (–0.01 %)	34.7354 (–0.04 %)	32.1665 (-0-.09 %)
Pakhare et al. [11] Shimpi et al., [26] Navier's method (SVSDBT)	39.4517 (0.00 %)	38.8299 (0.00 %)	37.0962 (–0.01 %)	34.7354 (–0.04 %)	32.1665 (–0.09 %)
Pakhare, et al. [11] Shimpi et al. [26] Navier's method TVSDBT	39.4517 (0.00 %)	38.8301 (0.00 %)	37.0981 (–0.01 %)	34.7431 (–0.02 %)	32.1847 (–0.03 %)
Touratier [10]	39.4517	38.8308	37.1009	34.7491	32.1948
EBBT (Timoshenko [8])	39.4719 (0.05 %)	39.3171 (1.25 %)	38.8446 (4.70 %)	38.0937 (9.63 %)	37.1120 (15.27 %)

Table 3 shows values of θ_n for $n = 3$ for the flexural vibration problem of the thick SVSDBT studied using SVIM. Table 3 shows that the SVIM results are identical with Navier results presented by Shimpi et al. [26] and Pakhare et al. [11] using SVSDBT, and closely agree with Navier method results presented by Shimpi et al. [26] and Pakhare et al. [11] using TVSDBT. The present SVIM results show negligible differences of 0 % for $l/h = 100$, –0.01 % for $l/h = 20$, –0.04 % for $l/h = 10$, –0.12 % for $l/h = 20/3$ and –0.26 % for $l/h = 5$ from the Touratier [10] exact results. The EBBT results as presented by Timoshenko [8] and shown in Table 3 is unsuitable for the eigenfrequency solution because of the solution error of more than 9.63 % for $l/h = 10$; and greater error for $l/h < 10$.

Table 3. Dimensionless free (natural) vibration frequencies θ_n of shear deformable thick simply supported beams for $\mu = 0.30$ and various values of l/h for $n = 3$

Reference	l/h				
	100	20	10	20/3	5
Present study (SVIM)	88.6914 (0.00 %)	85.6619 (–0.01 %)	78.1547 (–0.04 %)	69.5062 (–0.12 %)	61.4581 (–0.26 %)
Pakhare et al. [11] Shimpi et al. [26] Navier's method (SVSDBT)	88.6914 (0.00 %)	85.6619 (–0.01 %)	78.1547 (–0.04 %)	69.5062 (–0.12 %)	61.4581 (–0.26 %)
Pakhare, et al. [11] Shimpi et al. [26] Navier's method TVSDBT	88.6914 (0.00 %)	85.6634 (0.00 %)	78.1719 (-0.02 %)	69.5629 (-0.04 %)	61.5746 (–0.07 %)
Touratier [10]	88.6915	85.6671	78.1855	69.5908	61.6121
EBBT (Timoshenko [8])	88.7936 (0.12 %)	88.0158 (2.74 %)	85.7108 (9.63 %)	82.2414 (18.18 %)	78.0234 (26.62 %)

Table 4 shows values of θ_n for $n = 4$ for the flexural vibration of the thick SVSDBT studied using SVIM. Table 4 illustrates that the SVIM result for θ_n for $n = 4$ are identical with Navier results presented by Shimpi et al. [26] and Pakhare et al. [11] for thick SVSDBT beams, and closely agree with Navier method results presented by Shimpi et al. [26] and Pakhare et al. [11] for thick TVSDBT beams. The Table 4 further illustrate that the present SVIM results show insignificant differences of 0 % for $l/h = 100$; –0.01 % for $l/h = 20$; –0.09 % for $l/h = 10$,

–0.26 % for $l/h = 20/3$ and –0.56 % for $l/h = 5$ from the Touratier [10] exact results. Table 4 also illustrate the unsuitability of the EBBT results presented by Timoshenko [8] for the eigen solution because of the unacceptable errors of 4.70 % for $l/h < 20$ associated with the EBBT theory applied to the flexural vibration problems of thick beams for the fourth mode of vibration.

Table 4. Dimensionless free (natural) vibration frequencies θ_n of shear deformable thick simply supported beams for $\mu = 0.30$ and various values of l/h for $n = 4$

Reference	l/h				
	100	20	10	20/3	5
Present study (SVIM)	157.4877 (0.00 %)	148.3846 (–0.01 %)	128.6660 (–0.09 %)	109.2588 (–0.26 %)	93.2594 (–0.54 %)
Pakhare et al. [11] Shimpi et al. [26] Navier's method (SVSDBT)	157.4877 (0.00 %)	148.3846 (–0.01 %)	128.6660 (–0.09 %)	109.2588 (–0.26 %)	93.2594 (–0.54 %)
Pakhare, et al. [11] Shimpi et al. [26] Navier's method TVSDBT	157.4878 (0.00 %)	148.3924 (–0.01 %)	128.7389 (–0.03 %)	109.4660 (–0.07 %)	93.6436 (–0.13 %)
Touratier [10]	157.4882	148.4036	128.7792	109.5453	93.7660
EBBT (Timoshenko [8])	157.8099 (0.2 %)	155.3785 (4.70 %)	148.4480 (15.27 %)	138.7083 (26.62 %)	127.8170 (36.32 %)

The effectiveness and accuracy of the SVIM method for solving for the natural transverse vibration frequencies of thick beams using the SVSDBT has been demonstrated. The SVIM yields accurate results for the vibration modes which are in close agreement with the exact results of Touratier [10] and identical with Navier method results presented by Shimpi et al. [26] and Pakhare et al. [11] for thick beams modelled using SVSDBT and TVSDBT.

5. Conclusions

This study has derived a single variable shear deformable beam (SVSDB) partial differential equation of motion for free and forced transverse flexural vibrations. The work used first principles and then used the Stodola-Vianello iteration method for the first time, to present a novel systematic solution of the free vibration problem; for simply supported boundary conditions and for harmonic vibration.

In conclusion:

1) The SVSDB equation formulated using Shimpi et al. [12] displacement field assumptions led to a transverse shear stress quadratic distribution over the depth, and satisfies the transverse shear stress-free boundary conditions on the top and bottom surfaces.

2) The SVIM simplified the solution of the BVP to an algebraic iterative process which upon convergence yields an eigenequation for the eigenvalues.

3) The natural vibration frequencies for $n = 1, 2, 3$ and 4 obtained by SVIM are identical with the Navier method results previously obtained using various thick beam equations and are close to the exact Touratier [10] solutions.

4) The SVIM results obtained in this study are exact because exact eigenfunctions of simply supported beams are used in developing the Stodola-Vianello iteration formula that yielded the eigenvalues.

Acknowledgements

Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] C. C. Ike, "Fourier sine transform method for the free vibration of Euler-Bernoulli beam resting on Winkler foundation," *International Journal of Darshan Institute on Engineering Research and Emerging Technologies*, Vol. 7, No. 1, Jul. 2018, https://doi.org/10.32692/ijdi-eret/7.1.2018.1801

[2] C. C. Ike, "Timoshenko beam theory for the flexural analysis of moderately thick beams – variational formulation and closed form solutions," *Tecnica-Italiana – Italian Journal of Engineering Science*, Vol. 63, No. 1, pp. 34–45, 2019, https://doi.org/10.18280/ti-ijes/630105

[3] C. C. Ike, "Sumudu transform method for finding the transverse natural harmonic vibration frequencies of Euler-Bernoulli beams," *ARPN Journal of Engineering and Applied Sciences*, Vol. 16, No. 9, pp. 903–911, 2021.

[4] C. Ike, "Free vibration of thin beams on Winkler foundations using generalized integral transform method," *Engineering and Technology Journal*, pp. 1–12, Aug. 2023, https://doi.org/10.30684/etj.2023.140343.1462

[5] C. Ike and T. Elzaki, "Elzaki transform method for natural frequency analysis of Euler-Bernoulli beams," *Engineering and Technology Journal*, pp. 1–12, Aug. 2023, https://doi.org/10.30684/etj.2023.140211.1456

[6] S. Emadi, H. Ma, J. A. Lazano-Galant, and J. Turmo, "Simplified calculation of shear rotations for first-order shear deformation theory in deep bridge beams," *Applied Science*, Vol. 13, No. 5, p. 3362, 2023, https://doi.org/10.3390/app.13053362

[7] N. H. Erdelyi and S. M. Hashemi, "A dynamic stiffness element for free vibration analysis of delaminated layered beams," *Modelling and Simulation in Engineering*, Vol. 2012, pp. 1–8, Jan. 2012, https://doi.org/10.1155/2012/492415

[8] S. P. Timoshenko, "On the correction for shear of the differential equation for transverse vibrations of Prismatic bars," *Philosophical Magazine*, Vol. 41, pp. 744–746, 1921.

[9] C. R. Cowper, "On the accuracy of Timoshenko's beam theory," *ASCE Journal of Engineering Mechanics Division*, Vol. 94, No. 6, pp. 1447–1453, 1968.

[10] M. Touratier, "An efficient standard plate theory," *International Journal of Engineering Science*, Vol. 29, No. 8, pp. 901–916, 1991.

[11] K. S. Pakhare, P. J. Guruprasad, and P. Shimpi, "Effect of the variational consistency and variational inconsistency on free flexural vibration frequencies of simply supported rectangular isotropic shear deformable beams," in *International Conference on Materials, Mechanics and Structures 2020 (ICMMS 2020) IOP Conf. Series: Materials Science and Engineering*, Vol. 936, p. 012048, 2020, https://doi.org/10.1088/1757-899x/936/012048

[12] R. P. Shimpi, R. A. Shetty, and A. Guha, "A simple single variable shear deformation theory for a rectangular beam," *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, Vol. 231, No. 24, pp. 4576–4591, 2017.

[13] A. C. H. Vasconcelos, A. S. D. C. Az-Vêdo, and S. D. S. Hoefel, "Finite element analysis of shear deformation and rotatory inertia for beam vibration," in *Proceedings XXXVII Iberian Latin American Congress on Computational Methods in Engineering, Brazil, CILAMCE*, pp. 1–17, 2016.

[14] M. Levinson, "A new rectangular beam theory," *Journal of Sound and Vibration*, Vol. 74, No. 1, pp. 81–87, Jan. 1981, https://doi.org/10.1016/0022-460x(81)90493-4

[15] A. V. K. Murty, "On the shear deformation theory for dynamic analysis of beams," *Journal of Sound and Vibration*, Vol. 101, No. 1, pp. 1–12, 1985.

[16] G. Shi and G. Z. Voyiadjis, "A sixth-order theory of shear deformable beams with variational consistent boundary conditions," *ASME Journal of Applied Mechanics*, Vol. 78, No. 2, p. 021019, 2011.

[17] N. G. Stephen and M. Levinson, "A second order beam theory," *Journal of Sound and Vibration*, Vol. 67, No. 3, pp. 293–305, 1979.

[18] Y. M. Ghugal and R. Sharma, "Hyperbolic shear deformation theory for flexure and vibration of thick isotropic beams," *International Journal of Computational Methods*, Vol. 6, No. 4, pp. 585–604, Nov. 2011, https://doi.org/10.1142/s0219876209002017

[19] S. Thai, H.-T. Thai, T. P. Vo, and V. I. Patel, "A simple shear deformation theory for nonlocal beams," *Composite Structures*, Vol. 183, pp. 262–270, Jan. 2018, https://doi.org/10.1016/j.compstruct.2017.03.022

[20] A. Razouki, B. Lboucine, and E. B. Khalid, "The exact analytical solution of the bending analysis of thick functionally graded beams with higher order shear deformation theory using differential transform method," *International Journal of Advanced Research in Engineering and Technology*, Vol. 11, No. 5, pp. 194–203, 2020.

[21] H. Zioù, M. Guenfoud, and H. Guenfoud, "A simple higher order shear deformation theory for static bending analysis of functionally graded beams," *Jordan Journal of Civil Engineering*, Vol. 15, No. 2, pp. 209–209, 2021.

[22] P. R. Heyliger and J. N. Reddy, "A higher order beam finite element for bending and vibration problems," *Journal of Sound and Vibration*, Vol. 126, No. 2, pp. 309–326, Oct. 1988, https://doi.org/10.1016/0022-460x(88)90244-1

[23] Y. M. Ghugal and R. P. Shimpi, "A review of refined shear deformation theories for isotropic and anisotropic laminated beams," *Journal of Reinforced Plastics and Composites*, Vol. 21, pp. 775–813, 2002.

[24] A. S. Sayyad and Y. M. Ghugal, "Single variable refined beam theories for the bending, buckling and free vibration of homogeneous beams," *Applied and Computational Mechanics*, Vol. 10, No. 2016, pp. 123–138, 2016.

[25] O. M. Ibearugbulem, S. Sule, and C. O. Joeman, "Use of refined beam theory for free and forced vibration analysis of a deep prismatic beam," *Annals of Faculty Engineering, Hunedora – International Journal of Engineering*, Vol. XX, No. 2, pp. 13–18, 2022.

[26] R. P. Shimpi, P. J. Guruprasad, and K. S. Pakhare, "Simple two variable refined theory for shear deformable isotropic rectangular beams," *Journal of Applied and Computational Mechanics*, Vol. 6, No. Online First, pp. 394–415, Jul. 2019, https://doi.org/10.22055/jacm.2019.29555.1615

[27] A. S. Sayyad, "Comparison of various refined beam theories for the bending and free vibration analysis of thick beams," *Applied and Computational Mechanics*, Vol. 5, pp. 217–230, 2011.

[28] S. B. Dong, J. A. Wolf, and F. E. Peterson, "On a direct-iterative eigensolution technique," *International Journal for Numerical Methods in Engineering*, Vol. 4, No. 2, pp. 155–161, Jun. 2005, https://doi.org/10.1002/nme.1620040202

[29] M. Chennit, A. Ahmed-Chaouch, M. Saidani, and A. Bourzam, "Periods and mode shapes for uniform shear wall buildings: importance of selecting the appropriate dynamic behavior," *International Journal of Structural Stability and Dynamics*, Vol. 22, No. 11, p. 2250114, Sep. 2022, https://doi.org/10.1142/s0219455422501140

[30] C. C. Ike, "Stodola-Vianello method for buckling load analysis of Euler-Bernoulli beam on Winkler foundation," *UNIZIK Journal of Engineering and Applied Sciences*, Vol. 2, No. 1, pp. 250–259, 2023.

[31] C. C. Ike, "Stodola-Vianello methods for the buckling load analysis of Euler-Bernoulli beam on Pasternak foundation," *UNIZIK Journal of Engineering and Applied Sciences*, Vol. 2, No. 1, pp. 217–226, 2023.

[32] I. O. Ofondu, E. U. Ikwueze, and C. C. Ike, "Determination of the critical buckling loads of Euler columns using Stodola-Vianello iteration method," *Malaysian Journal of Civil Engineering*, Vol. 30, No. 3, pp. 378–394, 2018.

[33] C. C. Ike, "Critical buckling load solution of thin beam on Winkler foundation via polynomial shape function in Stodola-Vianello iteration method," *Journal of Research in Engineering and Applied Sciences*, Vol. 8, No. 3, pp. 591–595, 2023, https://doi.org/https:doi.org/10.46565/jreas.2023.83591-595

[34] C. C. Ike, A. O. Oguaghamba, and J. N. Ugwu, "Stodola-Vianello iteration method for the critical buckling load analysis of thin beam on Winkler foundation with clamped ends," in *Proceedings NIEEE Nsukka Chapter 4th Engineering Conference 2023*, pp. 34–38, 2023.

[35] C. C. Ike, A. O. Oguaghamba, and J. N. Ugwu, "Stodola-Vianello iteration method for the critical buckling load analysis of thin beam on two-parameter foundation with clamped ends," in *Proceedings Nigerian Institute of Electrical and Electronic Engineering (NIEEE) Nsukka Chapter 4th Engineering Conference*, pp. 1–5, 2023.

[36] C. C. Ike, "Eigenvalue solutions for Euler-Bernoulli beams on two-parameter foundations using Stodola-Vianello iteration method and polynomial basis functions," *Nnamdi Azikiwe University Journal of Civil Engineering*, Vol. 1, No. 4, pp. 59–68, 2023.

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