Fourier series method for the stability solution of simply supported thin beams on two-parameter elastic foundations of the Pasternak, Filonenko-Borodich, Hetenyi or Vlasov models

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Abstract. The analysis of stability problems of beams on two-parameter foundations (Bo2PFs) is an important part of their design for compressive loads. This work presents novel first principles derivation of the governing differential equations of elastic stability (GDES) of thin beams resting on two-parameter elastic foundations of the Pasternak, Filonenko-Borodich, Hetenyi or Vlasov models. The requirements of translational and rotational equilibrium of all the applied, reactive and internal forces on an infinitesimal segment of the Bo2PF and the laws of infinitesimal calculus were used to formulate the GDES as a fourth order ordinary differential equation (ODE) in terms of the transverse displacement function $u(x)$. The GDES is non-homogeneous in the presence of applied transverse load $q(x)$ but homogeneous when $q(x)$ vanishes. This study presents the Fourier series method (FSM) for solving the governing differential equation of stability (GDES) for the case of Dirichlet boundary conditions. The FSM has the advantage of amenability to differentiation, and integration due to the orthogonality properties of the sinusoidal functions. Implementation of the FSM by assuming the unknown function in the GDES as a Fourier series of infinite terms and the exploitation of orthogonalization simplifies the problem to an algebraic eigenvalue problem which is the characteristic buckling equation. The exact eigenvalues are found by algebraic solution of the buckling equation. The exact eigenvalues were used to find the exact buckling loads and the exact buckling load coefficients. The critical buckling load was found to correspond to the first buckling mode ($n = 1$), and is identical with previous solutions in the literature. Numerical calculations for the critical buckling load parameters K_{cr} were presented for the Bo2PF problem for values of the dimensionless foundation parameters $\bar{k}_1 = 0$, $\bar{k}_2 = 0$; $\bar{k}_1 = 100$, $\bar{k}_2 = 0$; $\bar{k}_1 = 0$, $\bar{k}_2 = 1$; $\bar{k}_1 = 100$, $\bar{k}_2 = 100$; $\bar{k}_1 = 0$, $\bar{k}_2 = 2.5$; $\bar{k}_1 = 100$, $\bar{k}_2 = 2.5$. The present solutions were compared with previous solutions for K_{cr} in the literature. The comparison shows that the present FSM results are identical with previous results obtained using various other methods such as Recursive Differentiation Method, Finite Element Method, Generalized Integral Transform Method (GITM) and Stodola-Vianello Iteration Method. The study has illustrated the effectiveness of the FSM for solving Bo2PFs.

Keywords: Fourier series method, Euler-Bernoulli beam theory, two-parameter elastic foundation, critical buckling load, governing differential equation of stability.

1. Introduction

Foundation beams have been modeled using beam on elastic foundation theory that incorporates the reaction of the elastic foundation in the governing equation of beam theory to produce a governing differential equation for the foundation beam. The effect of the elastic foundation has been shown to affect the resulting behaviour of the foundation beam in buckling, bending or vibration.

Euler-Bernoulli beam theory (EBBT) has been commonly used for thin beams where the depth

to span ratios are less than 0.05, while Timoshenko beam theory (TBT) and shear deformable beam theories of Dahake and Ghugal [1], Levinson [2] and Sayyad and Ghugal [3] have been used for thick beams where the depth to span ratios exceed 0.05. EBBT which assumes the Euler-Bernoulli Navier hypothesis of orthogonality of plane cross-sections to the longitudinal neutral axis before and after deformation disregards shear strains which are crucial in causing cross-sectional plane warping in beams. Hence, EBBT is suitable solely for thin beams, but produce unacceptable results for thick beams. EBBT equation is the fourth order ordinary differential equation in terms of the transverse deflection $u(x)$ and is given by Eq. (1):

$$EIu^{iv}(x) = q(x), \tag{1}$$

for beam bending under distributed transverse load distribution, $q(x)$.

This work is focused on thin beams and hence used EBBT.

Elastic foundations have been modeled using discrete parameter and continuously distributed parameter models. Continuously distributed parameter models use the theory of elasticity to derive the reaction forces on the beams from the soil while discrete parameter models rely on the use of one, two or a limited number of parameters to describe the soil reaction pressures on the beam. Continuously distributed parameter models are difficult to formulate and use and are sparsely used. Discrete parameter foundation models are commonly used due to ease of formulation, simplicity in use and simplicity in the nature of the resulting governing equation.

Commonly used discrete parameter foundation models are (i) Winkler model, which is a one-parameter model, [4], [5], [6] (ii) two-parameter models derived by Pasternak, Vlasov, Hetenyi and Filonenko-Borodich amongst others [7], [8], [9] and (iii) three-parameter models due to Kerr [10], [11] which are rarely used.

The one-parameter Winkler model, which is illustrated in Figure 1, assumes the supporting soil to be a system of vertical, closely spaced, independent, linear elastic Hookean springs whose stiffness is directly proportional to the deflection. Hence, in the Winkler model, the soil reaction $(r(x))$ on the beam is directly proportional to the beam deflection $(u(x))$ at the point, where the proportionality constant is the Winkler constant, k, which is used to characterise the soil [4], [5], [6]. Thus, in the Winkler model, the equation for the reaction on the beam is given by Eq. (2):

$$r(x) = ku(x). \tag{2}$$

The Winkler model fails to account for shear interaction of the springs and produces discontinuity issues.

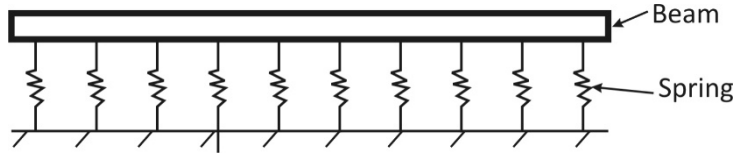

Fig. 1. Beam resting on a Winkler foundation model where the Winkler foundation model is a bed of closely speedy independent Hookean springs

The two-parameter foundation model, which is typically illustrated in Fig. 2, utilizes two-parameters to derive the soil reaction, in an effort to overcome the limitations of the Winkler model in disregarding the shear interaction of the springs. The two-parameter models thus introduce another foundation parameter to account for the shear interaction between the closely spaced, linear elastic, vertical Hookean springs [7], [8], [9]. Hence, while the first parameter (k_1) represents the soil stiffness in the vertical direction, the second parameter (k_2) represents the coupling/shear interaction effect of the vertical Hookean springs. The reaction $(r(x))$ for the two-parameter models have been previously derived using equilibrium and variational calculus

methods as expressed by Eq. (3):

$$r(x) = k_1 u(x) - k_2 \frac{d^2 u(x)}{dx^2}. \tag{3}$$

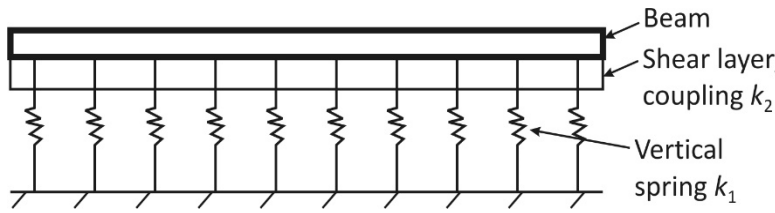

Fig. 2. Beam resting on a two-parameter foundation model where the two-parameter foundation model is graphically illustrated as a bed of closely spaced, linear elastic vertical Hookean springs with shear coupling present to model the shear interaction of the springs

Vlasov and Leontiev [12] adopted a simplified elastic continuum approach for the foundation and used the methods of variational calculus and energy minimization technique to derive the two-parameter Vlasov-Leontiev elastic foundation model. The Vlasov-Leontiev foundation model assumed the soil as an elastic layer and defined an arbitrary parameter γ_1 to characterize the vertical distribution of soil deformation.

Later, Jones and Xenophontos [13] developed an equation for the γ_1 parameter in terms of the displacement characteristics but did not provide any method for evaluating the γ_1 parameter. In another study, Vallabhan and Das [14], [15] determined the parameter γ_1 in terms of the beam and foundation soil characteristics using an iterative method and named the resulting model the modified Vlasov foundation.

Akhazhanov et al. [16] used the theory of elasticity to develop a simplified method for the analysis and solutions to beam on two-parameter elastic foundations. They validated their solution by comparison with solutions for beam on Winkler and Pasternak foundations and found that their solutions were identical with solutions based on Winkler and Pasternak foundation for various boundary conditions of the beam. Other contributions on beam on elastic foundations are in Akhazhanov et al. [17], [18], [19]; Huang et al. [20]. Akhmediev et al. [21] and Zhang et al. [22].

Gulkan and Alemdar [23] developed an analytical solution for the shape functions of a beam resting on a two-parameter elastic foundation. Their obtained solution is general and not restrictive to values of the foundation parameters. The exact shape function derived in their study is used to develop analytical expressions for the elements of the finite element stiffness matrices, nodal forces, mass and geometrical stiffness matrices.

Teodoru and Musat [24] also studied the finite element method of analysis of beam on linear variable two-parameter foundations. Olotu et al [25] used the semi-analytical approach called the differential transform method (DTM) to determine approximate solutions for the free vibration analysis of non-uniform beams resting on variable Winkler foundations. The elastic coefficients of the foundations were assumed to vary in the longitudinal dimensions of the beam. The DTM was applied to the equations of motion of the vibrating beam on variable Winkler foundation problem and the problem reduced to algebra. Computer codes implemented using MAPLE was used by Olotu et al. [25] to obtain solutions for clamped-clamped and simply supported boundaries, which were found to agree with previous solutions in the literature. Their works however did not consider buckling problems of beams on elastic foundation using DTM.

Aslami and Akimov [26] developed an efficient analytical method for the mathematical solution of continuous beams resting on two-parameter elastic foundations. The general form of their governing equation is reduced to a system of first order differential equations with constant coefficients; Jordan method in algebra is used to find the fundamental solution.

Beams on elastic foundations (BoEFs) subject to compressive forces can experience buckling failures, even when their material strengths have not been attained, when such compressive forces

attain a certain critical value called the critical buckling load. It is thus vital for their design to have a critical buckling assessment performed in order to determine the critical buckling loads that would result in buckling failures.

Hetenyi [27], Timoshenko and Gere [28] and Wang et al. [29] have developed governing differential equations for stability (GDES) of beam on elastic foundation (BoEF) and derived analytical solutions for such GDES under various boundary conditions (simply supported, clamped-clampled, clamped-simply supported and clamped-free). In recent studies, Taha [30] and Taha and Hadima [31] used recursive differentiation method (RDM) for solving the GDES for BoEF and obtained analytical solutions for critical buckling loads for non-prismatic BoEF.

Soltani [32] solved the GDES for BoEF using the numerical approach of finite element analysis. Hassan [33] and Aristizabal-Ochoa [34] presented solutions for BoEF under different end supports. Anghel and Mares [35] solved the GDES for BoEF by using collocation techniques to solve the resulting integral formulation of the GDES. Atay and Coskun [36] used the variational iteration method (VIM) to solve the GDES for BoEF for prismatic and non-prismatic beams.

Ike [4] used the finite sine transformation method (FSTM) for the free vibration analysis of prismatic beam on Winkler foundations (BoWF) and obtained exact eigenvalues. Ike [5] used the generalized integral transform method (GITM) for free vibration analysis of the BoWF for various boundary conditions. Ike [6] presented a point collocation method (PCM) for the approximate solutions of the bending problem of BoWF and obtained acceptable results, which compared well with previous results in the literature. Ike et al. [37] used Picard's iteration method for Euler buckling problems.

Ofondu et al. [38] used the Stodola-Vianello iteration method to solve Euler column buckling problems with clamped-pinned supports. Ikwueze et al. [39] used least squares weighted residual technique for the critical buckling load solutions of Euler columns with fixed-pinned ends. Mama et al. [40] used quintic polynomials as shape functions in a finite element analysis for the accurate buckling load solutions of BoWFs.

Ike et al [41] and Ike [42] used the Stodola-Vianello interaction method (SVIM) and polynomial shape function to find critical buckling load solutions of BoWF under clamped-clamped and simple end supports, respectively. Ike [43] applied the SVIM and exact trigonometric shape functions of simply supported beams to find exact critical buckling load solutions of BoWF for Dirichlet boundary conditions. Ike [7] implemented SVIM and exact trigonometric shape functions of Dirichlet boundary conditions to obtain exact eigenvalues for the eigenvalue problem of thin beam on Pasternak foundation. Ike et al. [8] and Ike [9] also implemented SVIM for Bo2PF using polynomial basis functions for clamped-clamped and simply supported boundaries, respectively.

Hariz et al. [44] investigated the buckling analysis of Timoshenko beam resting on two-parameter elastic foundations. Yue [45] used a refined beam model to study the behaviour of beams rested on two-parameter foundations by iterative method. Akgoz et al [46] studied the flexural analysis of beams on elastic foundations using the method of discrete singular convolution, but did not study buckling analysis.

Finite element methods were used in the studies of beam on elastic foundation by Alzubaidi et al. [47], Wieckowski and Swiatkiewiez [48], and Worku and Habte [49]. Theory of elasticity methods were also studied by Gholami and Alizadeh [50], Anyaegbunam [51] and Thanh and Linh [52]; but none studied buckling problems.

This work presents rigorous first principles derivation of the governing differential equations of elastic stability (GDES) for thin beams resting on two-parameter elastic foundations (Bo2PFs). The elastic foundations are modeled using Pasternak, Filonenko-Borodich, Hetenyi or Vlasov models using two-parameters, k_1 and k_2 while the slender beam is modeled using Euler-Bernoulli thin beam theory.

In this study, the Fourier series method (FSM) is adopted to obtain analytical solutions to simply supported thin beam resting on two-parameter foundations of the Pasternak, Hetenyi, Vlasov or Filonenko-Borodich type. The FSM is adopted due to its previously acclaimed success

in dealing with ordinary differential equations (ODEs) and partial differential equations (PDEs) with Dirichlet boundary conditions and due to its effectiveness as a mathematical tool in obtaining analytical solutions to boundary value problems (BVPs).

2. Novelty of the study

The novelty of the study is the first principles, systematic derivation of the GDES of the Bo2PF problem under axial compression load; and the systematic application of the FSM to the solution of the GDES for Dirichlet boundary conditions.

2.1. Governing differential equation of stability (GDES)

2.1.1. The thin beam on two-parameter foundation

Fig. 3 shows the thin beam resting on two-parameter elastic foundation which is considered in this study.

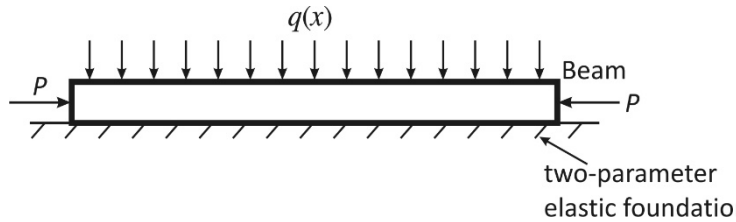

Fig. 3. Thin beam resting on two-parameter elastic foundation

Fig. 3 shows that the thin beam has a finite span and is subjected to axial compressive force P and transverse distributed load $q(x)$. The beam is simply supported at the ends $x = 0, x = l$ where l is the length of the beam.

2.1.2. Theoretical framework and assumptions

The assumptions of the formulation are as follows:
(i) The beam material is linearly elastic, homogeneous and isotropic.
(ii) The soil is linearly elastic, homogeneous and isotropic.
(iii) The displacements of the beam are very small relative to the thickness.
(iv) The axial strains are small as compared to unity.
(v) The normal strains in the transverse directions are so small and are considered insignificant.
(vi) The transverse shear stresses are also very small and negligible.
(vii) The cross-sections are plane and orthogonal to the longitudinal axis of the beam before and after flexural deformations.

2.2. Equations of equilibrium (EoE) for the elastic stability problem of Bo2PF

The free body diagram of an infinitesimal segment of the thin beam resting on two-parameter elastic foundation problem as considered in this paper is shown in Fig. 4.

In Fig. 4, P denotes the axial internal force at the left cut section; $P + \Delta P$ is the axial internal force at the right cut section. M is the bending moment acting at the left section, $M + \Delta M$ is the bending moment acting at the right section; $q(x)$ is the distributed transverse loading intensity, $r(x)$ is the reaction from the two-parameter foundation on the beam. The deflection of the beam within the infinitesimal length Δx is denoted by Δu. The elemental section of the thin beam on two-parameter elastic foundation is in equilibrium under the forces and reactions.

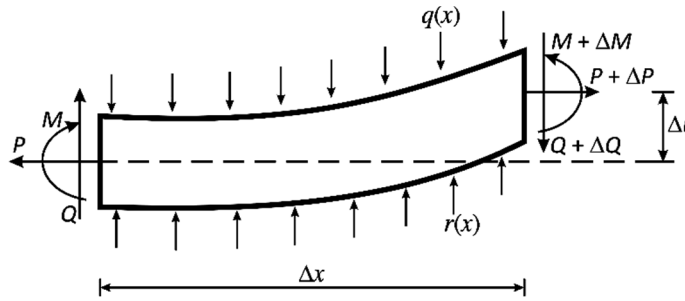

Fig. 4. Free body diagram of an elemental segment of a thin beam resting on a two-parameter elastic foundation under compressive force

For translational equilibrium in the axial (x) direction:

$$\sum F_{ix} = 0. \tag{4}$$

If forces in the positive x direction are assumed to be positive, the translational equilibrium equation becomes:

$$-P + (P + \Delta P) = 0. \tag{5}$$

Solving:

$$\Delta P = 0. \tag{6}$$

Hence, axial force, P is constant.
For translational equilibrium in the vertical direction:

$$\sum F_y = 0. \tag{7}$$

If downward forces are positive, then:

$$-Q + (Q + \Delta Q) + q(x)\Delta x - r(x)\Delta x = 0. \tag{8}$$

Simplifying:

$$\Delta Q + (q(x) - r(x))\Delta x = 0. \tag{9}$$

Dividing by Δx gives:

$$\frac{\Delta Q}{\Delta x} + q(x) - r(x) = 0. \tag{10}$$

In the limit as $\Delta x \to 0$:

$$\frac{dQ}{dx} = \lim_{\Delta x \to 0} \frac{\Delta Q}{\Delta x} = r(x) - q(x). \tag{11}$$

For rotational equilibrium about the left section, where anticlockwise moments are assumed as positive:

$$\sum M_i = 0. \tag{12}$$

If anticlockwise moments are positive, the rotational equilibrium equation is:

$$M + \Delta M - M - (P + \Delta P)\Delta u - (Q + \Delta Q)\Delta x - q(x)\frac{\Delta x \Delta x}{2} + r(x)\Delta x \frac{\Delta x}{2} = 0. \tag{13}$$

Simplifying, Eq. (13) gives:

$$\Delta M - P\Delta u - \Delta P \Delta u - Q\Delta x - \Delta Q \Delta x - q(x)\frac{(\Delta x)^2}{2} + r(x)\frac{(\Delta x)^2}{2} = 0. \tag{14}$$

Further simplification of Eq. (14) gives, from Eq. (6), the following:

$$\Delta M - P\Delta u - Q\Delta x - \Delta Q \Delta x - q(x)\frac{(\Delta x)^2}{2} + r(x)\frac{(\Delta x)^2}{2} = 0, \tag{15}$$

since $\Delta P = 0$.

Dividing Eq. (15) by Δx gives:

$$\frac{\Delta M(x)}{\Delta x} - P\frac{\Delta u(x)}{\Delta x} - Q - \Delta Q - q(x)\frac{\Delta x}{2} + r(x)\frac{\Delta x}{2} = 0. \tag{16}$$

In the limit as elemental quantities tend to zero, $\Delta x \to 0$, $\Delta Q \to 0$ and:

$$\lim\left(\frac{\Delta M(x)}{\Delta x} - P\frac{\Delta u}{\Delta x} - Q - \Delta Q - q(x)\frac{\Delta x}{2} + r(x)\frac{\Delta x}{2}\right) = 0. \tag{17}$$

Hence:

$$\frac{dM(x)}{dx} - P\frac{du(x)}{dx} - Q = 0. \tag{18}$$

Differentiating Eq. (18) with respect to x gives:

$$\frac{d}{dx}\left(\frac{dM}{dx} - P\frac{du}{dx} - Q\right) = 0. \tag{19}$$

Hence:

$$\frac{d^2M(x)}{dx^2} - P\frac{d^2u(x)}{dx^2} - \frac{dQ}{dx} = 0. \tag{20}$$

Substitution of Eq. (11) into Eq. (20) gives:

$$\frac{d^2M(x)}{dx^2} - P\frac{d^2u(x)}{dx^2} = \frac{dQ}{dx} = r(x) - q(x). \tag{21}$$

Hence:

$$\frac{d^2M(x)}{dx^2} - P\frac{d^2u(x)}{dx^2} = r(x) - q(x). \tag{22}$$

The moment deflection equation of Euler-Bernoulli beam theory is:

$$M(x) = -EIu''(x) = -EI\frac{d^2u(x)}{dx^2}. \tag{23}$$

Hence, the equation of equilibrium is:

$$\frac{d^2}{dx^2}\left(-EI\frac{d^2u(x)}{dx^2}\right) - P\frac{d^2u(x)}{dx^2} = r(x) - q(x). \tag{24}$$

The governing equation for the stability of thin prismatic beam on elastic foundations is given from equilibrium conditions as:

$$EIu^{iv}(x) + Pu''(x) = q(x) - r(x), \tag{25}$$

where the primes denote differentiations with respect to the space variable, x.

By substitution of the expression for $r(x)$ as given for two-parameter foundations by Eq. (2), the governing equation becomes:

$$EIu^{iv}(x) + Pu'' = q(x) - \left(k_1u(x) - k_2u''(x)\right). \tag{26}$$

Hence, simplifying gives:

$$EIu^{iv} + Pu''(x) + k_1u(x) - k_2u''(x) = q(x). \tag{27}$$

Hence, the governing equation is written explicitly as:

$$EI\frac{d^4u(x)}{dx^4} + P\frac{d^2u(x)}{dx^2} + k_1u(x) - k_2\frac{d^2u}{dx^2} = q(x), \tag{28}$$

where x is the axial coordinate axis, $u(x)$ is the deflection of the beam in the z direction, k_1 and k_2 are the two-parameters of the two-parameter foundation, I is the moment of inertia, E is the Young's modulus of elasticity of the beam material, P is the compressive load, $q(x)$ is the distribution of transverse load on the beam.

Dividing by EI gives:

$$\frac{d^4u}{dx^4} + \frac{P}{EI}\frac{d^2u}{dx^2} - \frac{k_2}{EI}\frac{d^2u}{dx^2} + \frac{k_1}{EI}u(x) = \frac{q(x)}{EI}, \tag{29}$$

$$u^{iv}(x) + \alpha u'' + \beta_2u''(x) + \beta_1u(x) = \frac{q(x)}{EI}, \tag{30}$$

where:

$$\alpha = \frac{P}{EI}, \quad \lambda_2 = \frac{k_2}{EI}, \quad \lambda_1 = \frac{k_1}{EI}. \tag{31}$$

When no transverse loading acts, $q(x) = 0$, and Eq. (9) becomes a homogeneous ODE:

$$u^{iv}(x) + (\alpha - \lambda_2)u''(x) + \lambda_1u(x) = 0. \tag{32}$$

3. Methodology

The Fourier series method assumes that the unknown displacement $u(x)$ is in the form of a

Fourier series given by:

$$u(x) = \sum_{n=1}^{\infty} \left(A_n \sin \frac{n\pi x}{l} + B_n \cos \frac{n\pi x}{l} \right), \tag{33}$$

where A_n and B_n are the coefficients of the Fourier series.

The thin beam considered is simply supported at the ends $x = 0$, and $x = l$.

The boundary conditions are:

$$u(x = 0) = 0, \tag{34a}$$
$$u(x = l) = 0. \tag{34b}$$

The force boundary conditions are:

$$M(x = 0) = 0, \quad M(x = l) = 0. \tag{35}$$

Hence from the bending moment deflection equation of Euler-Bernoulli theory, the force boundary conditions are expressed in terms of the displacement variable as:

$$u''(x = 0) = 0, \tag{36a}$$
$$u''(x = l) = 0. \tag{36b}$$

By differentiating $u(x)$ twice with respect to x, we have:

$$u''(x) = \sum_{n=1}^{\infty} -\left(\frac{n\pi}{l}\right)^2 \left(A_n \sin \frac{n\pi x}{l} + B_n \cos \frac{n\pi x}{l} \right). \tag{37}$$

Applying the boundary condition Eq. (34a) gives:

$$u(x = 0) = \sum_{n=1}^{\infty} (A_n \sin 0 + B_n \cos 0) = \sum_{n=1}^{\infty} B_n = 0. \tag{38}$$

Hence:

$$B_n = 0. \tag{39}$$

Applying the boundary condition Eq. (36a) gives:

$$u''(x = 0) = \sum_{n=1}^{\infty} -\left(\frac{n\pi}{l}\right)^2 (A_n \sin 0 + B_n \cos 0) = \sum_{n=1}^{\infty} -\left(\frac{n\pi}{l}\right)^2 B_n = 0. \tag{40}$$

Hence:

$$u(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}. \tag{41}$$

By differentiation of $u(x)$, we have:

$$u''(x) = \sum_{n=1}^{\infty} -\left(\frac{n\pi}{l}\right)^2 A_n \sin\frac{n\pi x}{l},$$
$$u^{iv}(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^4 A_n \sin\frac{n\pi x}{l}. \tag{42}$$

Then, substituting into the governing equation yields:

$$\sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^4 A_n \sin\frac{n\pi x}{l} - (\alpha - \lambda_2)\left(\frac{n\pi}{l}\right)^2 A_n \sin\frac{n\pi x}{l} + \lambda_1 A_n \sin\frac{n\pi x}{l} = 0. \tag{43}$$

Simplifying:

$$\sum_{n=1}^{\infty} A_n \left(\left(\frac{n\pi}{l}\right)^4 + (\lambda_2 - \alpha)\left(\frac{n\pi}{l}\right)^2 + \lambda_1\right) \sin\frac{n\pi x}{l} = 0. \tag{44}$$

Orthogonalizing by multiplication by $\sin\frac{m\pi x}{l}$ and integrating over the beam domain $(0 \le x \le l)$ gives:

$$\sum_{n=1}^{\infty} \int_0^l A_n \left(\left(\frac{n\pi}{l}\right)^4 + (\lambda_2 - \alpha)\left(\frac{n\pi}{l}\right)^2 + \lambda_1\right) \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} dx = 0. \tag{45}$$

Simplifying, Eq. (45) gives Eq. (46);

$$\sum_{n=1}^{\infty} A_n \left(\left(\frac{n\pi}{l}\right)^4 + (\lambda_2 - \alpha)\left(\frac{n\pi}{l}\right)^2 + \lambda_1\right) \int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} dx = 0. \tag{46}$$

Or:

$$\sum_{n=1}^{\infty} A_n \left(\left(\frac{n\pi}{l}\right)^4 + (\lambda_2 - \alpha)\left(\frac{n\pi}{l}\right)^2 + \lambda_1\right) I_{mn} = 0, \tag{47}$$

where:

$$I_{mn} = \int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\, dx. \tag{48}$$

The sine functions are orthogonal. Hence, when $m \neq n$, I_{mn} would vanish; but would not vanish when $m = n$. The orthogonality conditions are:

$$I_{mn} = \int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} dx = 0. \tag{48a}$$

if $m \neq n$ and non-zero if $m = n$:

$$I_{mn} = \int_0^l \sin^2 \frac{n\pi x}{l}\, dx. \tag{48b}$$

The orthogonality conditions of the sinusoidal functions simplify Eq. (46) and enable the determination of the eigensolutions.

4. Results

The characteristic buckling equation is given from the conditions for nontrivial solutions to Eq. (47) by Eq. (49):

$$\left(\frac{n\pi}{l}\right)^4 + (\lambda_2 - \alpha)\left(\frac{n\pi}{l}\right)^2 + \lambda_1 = 0. \tag{49}$$

Solving for α, by first simplifying Eq. (49) gives Eq. (50):

$$\left(\frac{n\pi}{l}\right)^4 + \lambda_1 = -(\lambda_2 - \alpha)\left(\frac{n\pi}{l}\right)^2. \tag{50}$$

Further simplification of Eq. (50) gives Eq. (50a):

$$\left(\frac{n\pi}{l}\right)^4 + \lambda_1 = (\alpha - \lambda_2)\left(\frac{n\pi}{l}\right)^2. \tag{50a}$$

Dividing Eq. (50a) by $\left(\frac{n\pi}{l}\right)^2$ gives Eq. (51):

$$\alpha - \lambda_2 = \frac{\left(\frac{n\pi}{l}\right)^4 + \lambda_1}{\left(\frac{n\pi}{l}\right)^2}. \tag{51}$$

Simplifying Eq. (51) yields:

$$\alpha - \lambda_2 = \left(\frac{l}{n\pi}\right)^2 \left(\lambda_1 + \left(\frac{n\pi}{l}\right)^4\right). \tag{52}$$

Hence:

$$\alpha = \lambda_2 + \left(\frac{l}{n\pi}\right)^2 \left(\lambda_1 + \left(\frac{n\pi}{l}\right)^4\right). \tag{53}$$

Or:

$$\alpha = \lambda_2 + \left(\frac{\lambda_1 l^2}{(n\pi)^2} + \left(\frac{n\pi}{l}\right)^2\right). \tag{53a}$$

Using Eq. (31) in Eq. (53a) gives:

$$P = EI\alpha = EI\left(\lambda_2 + \frac{\lambda_1 l^2}{(n\pi)^2} + \left(\frac{n\pi}{l}\right)^2\right). \tag{54}$$

Expressing Eq. (54) in the standard form gives Eq. (55):

$$P = \frac{EI}{l^2}\left(\lambda_2 + \frac{\lambda_1 l^2}{(n\pi)^2} + \left(\frac{n\pi}{l}\right)^2\right) l^2. \tag{55}$$

Simplifying Eq. (55) gives Eq. (55a):

$$P = \frac{EI}{l^2}\left(\lambda_2 l^2 + \frac{\lambda_1 l^4}{(n\pi)^2} + (n\pi)^2\right). \tag{55a}$$

Hence introducing the dimensionless buckling coefficient $K(\lambda_1, \lambda_2)$ gives:

$$P = \frac{EI}{l^2} K(\lambda_1, \lambda_2), \tag{56}$$

where:

$$K(\lambda_1, \lambda_2) = \lambda_2 l^2 + \frac{\lambda_1 l^4}{(n\pi)^2} + (n\pi)^2. \tag{57}$$

The critical buckling load coefficient K_{cr} is the value of the least K_{cr}, and this occurs at the first buckling mode where $n = 1$. Thus:

$$K_{cr} = K(\lambda_1, \lambda_2, n = 1) = \lambda_2 l^2 + \frac{\lambda_1 l^4}{\pi^2} + \pi^2. \tag{58}$$

Then, the critical buckling load P_{cr} which is associated with K_{cr} is found as:

$$P_{cr} = \frac{EI}{l^2} K_{cr}. \tag{59}$$

Values of K_{cr} are calculated for certain nondimensional values of $\bar{k}_1$ and $\bar{k}_2$ expressed in terms of λ_1, λ_2 and l and given in Table 1.

Table 1. Critical buckling load parameters square root of K_{cr} for thin beam on two-parameter elastic foundation for various values of the elastic foundation parameters

$\bar{k}_2 = \lambda_2\left(\frac{l}{\pi}\right)^2 = 0$					
$\bar{k}_1 = \lambda_1 l^4$	Taha [30]	Anghel and Mares [35]	Ike [7]	Ike [9]	Present
0	3.1415	3.1413	3.141593	3.143621	3.141593
100	4.4723	4.4721	4.472329	4.473597	4.472329
$\bar{k}_2 = \lambda_2\left(\frac{l}{\pi}\right)^2 = 1$					
$\bar{k}_1 = \lambda_1 l^4$	Taha [30]	Anghel and Mares [35]	Ike [7]	Ike [9]	Present
0	4.4428	4.4427	4.44283	4.444317	4.44283
100	5.4654	5.4653	5.465467	5.466505	5.465467
$\bar{k}_2 = \lambda_2\left(\frac{l}{\pi}\right)^2 = 2.5\frac{1}{2}$					
$\bar{k}_1 = \lambda_1 l^4$	Taha [30]	Anghel and Mares [35]	Ike [7]	Ike [9]	Present
0	5.8774	5.8772	5.877382	5.878466	5.877382
100	6.6840	6.6838	6.683991	6.68484	6.683991

5. Discussion

This work has presented a first principles systematic derivation of the GDES for Bo2PFs. The GDES was derived using equilibrium Newtonian approach, and considered both translational and rotational equilibrium of an infinitesimal Bo2PF. The GDES is found as an ODE of fourth order.

This study has applied the Fourier series method to develop exact buckling solutions for simply supported Euler-Bernoulli beam resting on two-parameter elastic foundations of the Pasternak, Hetenyi, Vlasov or Filonenko-Borodich type. The GDES is a fourth order ODE which is non-homogeneous when transverse load is present, but homogeneous when transverse load vanishes, $q(x) = 0$.

By assuming the unknown function $u(x)$ in the form of a Fourier series of unknown coefficients, and applying orthogonality of the Fourier series the GDES is reduced to a characteristic buckling equation given by the algebraic Eq. (49). The eigenvalue is found from the roots of Eq. (49) as Eq. (53) yielding the buckling load for any buckling mode, n, as Eq. (54).

Critical buckling load coefficient K_{cr} is found to correspond to the first buckling mode $(n = 1)$ and given by Eq. (58). Eq. (58) is presented in Table 1 for various values of the elastic foundation's dimensionless two-parameters $\bar{k}_1$ and $\bar{k}_2$, along with previous values of K_{cr} determined by Taha [30], Anghel and Mares [35] and Ike [7], [9]. Table 1 illustrates that present FSM results are identical with the results obtained by Ike [7] by SVIM and using exact buckling shape functions and Ike [9] using fourth degree polynomial shape foundation in SVIM; Taha [30] results found by RDM, and Angel and Mares [35] results by Collocation method. The obtained solutions are exact within the framework of the Euler-Bernoulli beam stability equation considered because exact buckling shape functions were used and the solutions satisfied the domain equations at all solution points.

6. Conclusions

This study has presented first principles derivation using Newtonian equilibrium method for the elastic stability problem of Bo2PF.

This work has studied the FSM for solving the GDES for Bo2PFs. In conclusion:

1) The Fourier series method gives an exact analytical solution to the eigenvalue problem of simply supported beam on two-parameter foundations of the Pasternak, Hetenyi, Vlasov or Filenenko-Borodich type.

2) The critical buckling load occurs at the first buckling mode.

3) The critical buckling load expression is the exact analytical expression because the Fourier series contains the exact buckling shape function for the Dirichlet boundary conditions of the Bo2PF considered.

4) The results obtained for the critical buckling load parameters are identical with previous solutions that used SVIM, RDM and Collocation.

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Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] A. G. Dahake and Y. M. Ghugal, "A trigonometric shear deformation theory for flexure of thick beam," *Procedia Engineering*, Vol. 51, pp. 1–7, Jan. 2013, https://doi.org/10.1016/j.proeng.2013.01.004

[2] M. Levinson, "A new rectangular beam theory," *Journal of Sound and Vibration*, Vol. 74, No. 1, pp. 81–87, Jan. 1981, https://doi.org/10.1016/0022-460x(81)90493-4

[3] A. S. Sayyad and Y. M. Ghugal, "Flexure of thick beams using new hyperbolic shear deformation theory," *International Journal of Mechanics*, Vol. 5, No. 3, pp. 113–122, 2011.

[4] C. C. Ike, "Fourier Sine Transform method for the free vibration of Euler-Bernoulli beam resting on Winkler foundation," *International Journal of Darshan Institute on Engineering Research and Emerging Technologies*, Vol. 7, No. 1, Jul. 2018, https://doi.org/10.32692/ijdi-eret/7.1.2018.1801

[5] C. Ike, "Free vibration of thin beams on Winkler foundations using generalized integral transform method," *Engineering and Technology Journal*, Vol. 41, No. 11, pp. 1286–1297, Aug. 2023, https://doi.org/10.30684/etj.2023.140343.1462

[6] Ike Charles, "Point collocation method for the analysis of Euler-Bernoulli beam on Winkler foundation," *International Journal of Darshan Institute on Engineering Research and Emerging Technology*, Vol. 7, No. 2, Jan. 2019.

[7] Ike C. C., "Stodola-Vianello method for the buckling load analysis of Euler-Bernoulli beam on Pasternak Foundation," *UNIZIK Journal of Engineering and Applied Sciences*, Vol. 2, No. 1, pp. 217–226, Jun. 2023.

[8] C. C. Ike, O. A. Oguaghamba, and J. N. Ugwu, "Stodola-Vianello iteration method for the critical buckling load analysis of thin beam on two-parameter foundation with clamped ends," in *Proceedings Nigerian Institute of Electrical and Electronic Engineers (NIEEE) Nsukka Chapter*, 2023.

[9] C. C. Ike, "Eigenvalue solutions for Euler-Bernoulli beams on two-parameter foundations using Stodola-Vianello iteration method and polynomial basis functions," *Nnamdi Azikiwe University Journal of Civil Engineering*, Vol. 1, No. 4, pp. 59–68, 2023.

[10] A. D. Kerr, "Elastic end visco-elastic foundation models," *Journal of Applied Mechanics*, Vol. 31, No. 3, pp. 491–498, 1964, https://doi.org/10.1007/s11071-019-04977-9

[11] A. D. Kerr, "On the determination of foundation model parameters," *Journal of Geotechnical Engineering*, Vol. 111, No. 11, pp. 1334–1340, Nov. 1985, https://doi.org/10.1061/(asce)0733-9410(1985)111:11(1334)

[12] V. Z. Vlasov and N. N. Leontiev, "Beams, plates and shells on elastic foundation," Israel Program for Scientific Translations, 1966.

[13] R. Jones and J. Xenophontos, "The Vlasov foundation model," *International Journal of Mechanical Sciences*, Vol. 19, No. 6, pp. 317–323, Jan. 1977, https://doi.org/10.1016/0020-7403(77)90084-4

[14] C. V. G. Vallabhan and Y. C. Das, "Parametric study of beams on elastic foundation," *Journal of Geotechnical Engineering*, Vol. 114, No. 12, pp. 2072–2082, 1988.

[15] C. V. G. Vallabhan and Y. C. Das, "Modified Vlasov method for beams on elastic foundations," *Journal of Geotechnical Engineering*, Vol. 117, No. 6, pp. 956–966, 1991.

[16] S. Akhazhanov, B. Bostanov, A. Kaliyev, T. Akhazhanov, and A. Mergenbekova, "Simplified method of calculating a beam on a two-parameter elastic foundation," *International Journal of Geomate*, Vol. 25, No. 111, pp. 33–40, Nov. 2023, https://doi.org/10.2166/2003.111.3898

[17] S. Akhazhanov, N. Vatin, S. Akhmediyev, T. Akhazhanov, O. Khabidolda, and A. Nurgoziyeva, "Beam on a two-parameter elastic foundation: Simplified finite element model," *Magazine of Civil Engineering*, Vol. 121, No. 5, pp. 12107–12107, Jan. 2023, https://doi.org/10.34910/mce.121.7

[18] S. Akhazhanov, "Analytical solution of beams on elastic foundation," *International Journal of GEOMATE*, Vol. 19, No. 73, pp. 277–299, Sep. 2020, https://doi.org/10.21660/2020.73.51487

[19] S. Akhazhanov, D. Baltabai, and B. Nurlanova, "Refined mechanical and mathematical model of an elastic half-plane," *Technobius*, Vol. 2, No. 1, Mar. 2022, https://doi.org/10.54355/tbus/2.1.2022.0014

[20] M. Huang, X. Zhou, J. Yu, C. F. Leung, and J. Q. W. Tan, "Estimating the effects of tunnelling on existing jointed pipelines based on Winkler model," *Tunnelling and Underground Space Technology*, Vol. 86, pp. 89–99, Apr. 2019, https://doi.org/10.1016/j.tust.2019.01.015

[21] S. Akhmediev, G. Mikhailov Tazhenova, M. Bakirov, T. Filippova, and D. Tokanov, "Calculating a beam of variable section lying on an elastic foundation," *Journal of Applied Engineering Science*, Vol. 21, No. 1, pp. 87–93, 2023.

[22] J. M. Zhang, Y. J. Shang, and R. Y. Wang, "Force analysis method of buried pipeline in landslide section based on Pasternak double parameter model," *Journal of Central South University (Science*

and Technology Edition), Vol. 51, No. 5, pp. 1328–1336, 2020, https://doi.org/10.1007/s10706-021-01975-w.pdf

[23] P. Gulkan and B. N. Alemdar, "An exact finite element for a beam on a two-parameter elastic foundation: a revisit," *Structural Engineering and Mechanics*, Vol. 7, No. 3, pp. 259–276, Mar. 1999, https://doi.org/10.12989/sem.1999.7.3.259

[24] I. B. Teodora and V. Musat, "Beam elements on linear variable two-parameter elastic foundation," *Bulletinul Institutului Politehnic dini lasi, LIV(LVIII) Fasc2*, Vol. 2008, 2008.

[25] O. T. Olotu, O. O. Agboola, and J. A. Gbadeyan, "Free vibration analysis of non-uniform Rayleigh beams on variable Winkler elastic foundation using differential transform method," *Ilorin Journal of Science*, Vol. 8, No. 1, pp. 1–20, Jun. 2021, https://doi.org/10.54908/iljs.2021.08.01.001

[26] M. Aslami and P. A. Akimov, "Analytical solution for beams with multipoint boundary conditions on two-parameter elastic foundations," *Archives of Civil and Mechanical Engineering*, Vol. 16, No. 4, pp. 668–677, Sep. 2016, https://doi.org/10.1016/j.acme.2016.04.005

[27] M. Hetenyi, "Beams on elastic foundation: Theory with applications in the field of Civil and Mechanical Engineering," Ann Arbor, 1946.

[28] S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*. McGraw-Hill International Book Company, 1985.

[29] C. M. Wang and C. Y. Wang, *Exact Solutions for Buckling of Structural Members*. CRC Press, 2005.

[30] M. H. Taha, "Recursive differentiation method of beams for boundary value problems: application to analysis of a beam-column on an elastic foundation," *Journal of Theoretical and Applied Mechanics Sofia*, Vol. 44, No. 2, pp. 57–70, 2014.

[31] M. H. Taha and S. A. Hadima, "Analysis of non-uniform beams on elastic foundations using recursive differentiation method," *Engineering Mechanics*, Vol. 22, No. 83-94, 2015.

[32] M. Soltani, "Finite element modeling for buckling analysis of tapered axially functionally graded timoshenko beam on elastic foundation," *Mechanics of Advanced Composite Structures*, Vol. 7, No. 2, pp. 203–218, Nov. 2020, https://doi.org/10.22075/macs.2020.18591.1223

[33] M. Hassan, "Buckling of beams on elastic foundations," *Al Rafidain Engineering Journal*, Vol. 16, pp. 104–122, 2008, https://doi.org/10.33899/rengj.20.44659

[34] Dario Aristizabal-Ochoa, "Stability of slender columns on an elastic foundation with generalized end conditions," *Ingeniería e Investigación*, Vol. 33, No. 3, pp. 34–40, 2013.

[35] V. Anghel and C. Mares, "Integral formulation for stability and vibration analysis of beams on elastic foundation," *Proceedings of the Romanian Academy Series A*, Vol. 20, No. 3, pp. 285–293, 2019.

[36] M. T. Atay and S. B. Coşkun, "Elastic stability of Euler columns with a continuous elastic restraint using variational iteration method," *Computers and Mathematics with Applications*, Vol. 58, No. 11-12, pp. 2528–2534, Dec. 2009, https://doi.org/10.1016/j.camwa.2009.03.051

[37] C. C. Ike, E. U. Ikwueze, and I. O. Ofondu, "Picard's successive iteration method for the elastic buckling analysis of Euler columns with pinned ends," *Saudi Journal of Civil Engineering*, Vol. 2, No. 2, pp. 76–88, 2018.

[38] Ofondu I. O., Ikwueze E. U., and Ike C. C., "Determination of the critical buckling loads of Euler columns using Stodola-Vianello iteration method," *Malaysian Journal of Civil Engineering*, Vol. 30, No. 3, pp. 378–394, Nov. 2018, https://doi.org/10.11113/mjce.v30.16064

[39] E. U. Ikwueze, C. C. Ike, and I. O. Ofondu, "Least squares weighted residual method for the elastic buckling analysis of Euler column with fixed pinned ends," *Saudi Journal of Civil Engineering*, Vol. 2, No. 2, pp. 110–119, 2018.

[40] B. O. Mama, O. A. Oguaghamba, and C. C. Ike, "Quintic polynomial shape functions for the finite element analysis of elastic buckling loads of Euler-Bernoulli beams resting on Winkler foundation," in *Proceedings 2nd Nigerian Institute of Electrical and Electronic Engineers Nsukka Chapter Conference*, 2020.

[41] Charles Chinwuba Ike, Aloysius Onyedikachi Oguaghamba, and Ugwu Juliet, "Stodola-Vianello iteration method for the critical buckling load analysis of thin beam on Winkler foundation with clamped end," in *NIEEE Nsukka chapter 4th National Engineering Conference on Advancement for Sustainable Development in Developing Nations*, Aug. 2023.

[42] Charles Chinwuba Ike, "Critical buckling load solution of thin beam on Winkler foundation via polynomial shape function in Stodola-Vianello iteration method," *Journal of Research in Engineering and Applied Sciences*, Vol. 8, No. 3, pp. 591–595, Aug. 2023, https://doi.org/10.46565/jreas.202383591-596

[43] Charles Chinwuba Ike, "Stodola-Vianello method for the buckling load analysis of Euler-Bernoulli beam on Winkler foundation," *UNIZIK Journal of Engineering and Applied Sciences*, Vol. 2, No. 1, pp. 250–259, Jun. 2023.

[44] M. Hariz, L. Le Marrec, and J. Lerbet, "Buckling of Timoshenko beam under two-parameter elastic foundations," *International Journal of Solids and Structures*, Vol. 244-245, p. 111583, Jun. 2022, https://doi.org/10.1016/j.ijsolstr.2022.111583

[45] F. Yue, "A refined model for analysis of beams on two-parameter foundations by iterative method," *Mathematical Problems in Engineering*, Vol. 2021, pp. 1–11, Apr. 2021, https://doi.org/10.1155/2021/5562212

[46] B. Akgoz, K. Mercan, C. Demir, and O. Civalék, "Static analysis of beams on elastic foundation by the method of discrete singular convolution," *International Journal of Engineering and Applied Sciences*, Vol. 8, No. 3, pp. 67–73, 2016.

[47] R. Alzubaidi, H. M. Husain, and S. Shukur, "Finite element analysis of beam resting on footing," *International Journal of Geomate*, Vol. 24, No. 105, pp. 26–32, 2023.

[48] Z. Więckowski and P. Świątkiewicz, "Stress-based FEM in the problem of bending of Euler-Bernoulli and Timoshenko beams resting on elastic foundation," *Materials*, Vol. 14, No. 2, p. 460, Jan. 2021, https://doi.org/10.3390/ma14020460

[49] A. Worku and B. Habte, "Analytical formulation and finite-element implementation technique of a rigorous two-parameter foundation model to beams on elastic foundations," *Geomechanics and Geoengineering*, Vol. 17, No. 2, pp. 547–560, Mar. 2022, https://doi.org/10.1080/17486025.2020.1827162

[50] M. Gholami and M. Alizadeh, "A quasi 3D modified strain gradient formulation for static bending of functionally graded micro beams resting on Winkler-Pasternak elastic foundation," *Scientia Iranica*, Vol. 29, No. 1, pp. 26–40, May 2021, https://doi.org/10.24200/sci.2021.55000.4019

[51] A. J. Anyaegbunam, "Complete stresses and displacements in a cross-anisotropic half-space caused by a surface vertical point load," *International Journal of Geomechanics*, Vol. 14, No. 2, pp. 171–181, 2014.

[52] L. T. Thanh and H. N. T. Linh, "Problem solution of the elasticity theory for the half-plane," in *Journal of Physics: Conference Series*, Vol. 1803, No. 1, p. 012002, Feb. 2021, https://doi.org/10.1088/1742-6596/1803/1/012002

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