A novel problem and algorithm for solving permuted cordial labeling of corona product between two graphs

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Abstract. This study has come up with a new application of permuted cordial labeling initiated by two graphs based on their corona product, furthering the cause of a better comprehension of and research into specific types of graphs. The Permuted cordial labeling construction for the corona product of graphs consisting of paths, cycles, second power of a path and second power of cycle graphs may facilitate the consideration of the properties and structures of the graphs. It helps us to study its topological properties, connectivity images, symmetries and other properties.

Keywords: graph theory, labeling, machine learning, algorithms, network security, problem-solving.

1. Introduction

Graph theory is widely known to have applications in many other academic fields, including communication, physics, chemistry, biology, computer science, psychology, sociology, and economics [1, 2]. One area of graph theory that has received a lot of recent development is graph labelling. Labeled graphs are useful models for many fields, including as coding theory, circuit design, radar, X-ray crystallography, and database management [3].

A particular kind of graph labeling involves assigning values from an established set to its vertices, assigning an automatically induced label to its edges, and ensuring that the labeling satisfies certain conditions. On this subject, the paper by Gallian [4] is an excellent resource. Two of the most important categories are labeling that is graceful and harmonic. In 1966 and 1972, respectively, Rosa [5] and Golomb [6] separately developed graceful labeling, and in 1980, Graham and Sloane [7] concluded the first investigation on harmonic labeling.

A third important type of labeling that Cahit [8] developed in 1990 and which combines components of the other two is cordial labeling. Cordial labelings use labels 0 and 1, together with the induced label $(f(v)+f(w))(mod2)$, which naturally equals $|f(v)-f(w)|$, in contrast to graceful and harmonious labelings, which use labels $|f(v)-f(w)|$ and $f(v)+f(w)$ (modulo the number of edges), respectively. Since mathematics modulo 2 is an essential part of computer science, cordial labeling has a lot in common with that field. In [9] A. Abd El-hay and et.al, prove the corona product of paths and third power of lemniscate graph $P_k \odot L_{n,m}^3$ are signed product-cordial for all k, n and m. In [10] ELrokh and et.al, proved that each path, cycle and Fan admits permuted cordial labeling. The Wheel graph W_n, $n \geq 3$ admits permuted cordial labeling except $n \equiv 2mod3$ and n even. Moreover, they proved that the union of $P_n \cup P_m$, n, $m \geq 2$ admits a permuted cordial labeling for all n, m. The union of $C_n \cup C_m$, n, $m \geq 3$ admits a permuted cordial labeling for all n, m. The union of $P_n \cup C_m$, $n \geq 2$, $m \geq 3$ admits a permuted cordial labeling for all n, m. For more details about other labeling that are related to cordial labeling such as permuted logically, total cordial, signed product cordial, etc. the reader is referred to [10-17].

The rest of this paper is structured as follows: permuted cordial labeling for the corona product of paths, cycles, second power of paths, and second power of cycles are presented in Section 3.

Section 4 proposes an algorithm for determining the permuted cordial labeling of a given graph. Section 5 investigate the discussion on Permuted Cordial Labeling and Its Engineering Applications. Finally, in Section 6, conclusions are drawn.

2. Materials and methods

We can use these code of labeling as follows.

Table 1. Labeling methods

D_{3r}	$gfi \ldots . gfi$ repeated r time	W_{3k}	$gfi \ldots . gfi$ repeated k time
W_{3r}	$fgi \ldots . fgi$ repeated r time	E_{3k}	$fgi \ldots . fgi$ repeated k time
M_{3r}	$igf \ldots . igf$ repeated r time	M_{3k}	$igf \ldots . igf$ repeated k time
E_{3r}	$gif \ldots . gif$ repeated r time	D_{3k}	$gif \ldots . gif$ repeated k time
P_{3r}	$ifg \ldots . ifg$ repeated r time	S_{3k}	$ifg \ldots . ifg$ repeated k time
S_{3r}	$fig \ldots . fig$ repeated r time	P_{3k}	$fig \ldots . fig$ repeated k time

Sometimes, we modify D_{3r}, W_{3r}, and M_{3r} by adding symbols at one end or the other or both; for example, $M_{3r}gI$ means the labeling $Igf \ldots Igf(r\text{-times})gi$. Similarly, $M_{3r}f$ means the labeling $Igf \ldots Igf(r\text{-time})f$. $v(I)$, $v(g)$, and $v(f)$ represent the number of vertices labeled I, g, and f, respectively. Similarly, we denoted $e(i)$, $e(g)$, and $e(f)$ to be the number of edges labeled I, g, and f, respectively, for the graph G.

A vertex labeling of graph G of $h: V \to \{I, g, f\}$ with induced edge labeling $h^*: E(G) \to \{I, g, f\}$ defined by:

$$\begin{array}{cccc} \circ & v(i) & v(f) & v(g) \\ u(i) & I & f & g \\ u(f) & f & g & I \\ u(g) & g & I & f \end{array} \tag{1}$$

is called permuted cordial labeling if $|v(x) - v(y)| \leq 1$ and $|e(x) - e(y)| \leq 1$, $x \neq y$ and $x, y \in \{I, f, g\}$ where $v(x)$ (respectively, $e(x)$) is the number of vertices (respectively, edges) labeled with $x \in \{I, f, g\}$. A graph G is permuted cordial if it admits a permuted labeling. For a given labeling of the corona $G \odot H$, we denote $v(j)$ and $e(j)$ $(j = I, f, g)$ to represent the numbers of vertices and edges labeled by j, respectively. We denote x_j and a_j to be the numbers of vertices and edges labeled by $\{I, f, g\}$, respectively, for the graph G. Also, we let y_j, y_i' and b_j, b_i' be those for H, which are connected to the vertices labeled j of G. It is easy to verify that:

$$\begin{cases} v_i = x_i + x_i y_i + x_f y'_i + x_g y''_i, \\ v_f = x_f + x_i y_f + x_f y'_f + x_g y''_f, \\ v_g = x_g + x_i y_g + x_f y'_g + x_g y''_g, \\ e_i = a_i + x_i b_i + x_f b'_i + x_g b''_i + x_i y_i + x_f y'_g + x_g y''_f, \\ e_f = a_f + x_i b_f + x_f b'_f + x_g b''_f + x_i y_f + x_f y'_i + x_g y''_g, \\ e_g = a_g + x_i b_g + x_f b'_g + x_g b''_g + x_i y_g + x_f y'_f + x_g y''_i. \end{cases} \tag{2}$$

Finally, for particular labeling A and B that are used for G_1 and G_2 , respectively, we let $[A; B]$ denote the labeling for the corona of G_1 and G_2. Section one contains a brief literary analysis of the topic of this work. Section Two deals with the Materials and Methods employed throughout. Whereas section three is devoted to studying the permuted cordial labeling for the corona product of paths, cycles, second power of paths, and second power of cycles. Section four proposes an algorithm for determining the permuted cordiality of a given graph. Section five investigated the discussion on Permuted Cordial Labeling and Its Engineering Applications. The last section is the conclusion which summarizes the important points of our finding in this paper.

3. Results

In this section we shall prove that the permuted cordial labeling for the corona product of paths, cycles, second power of paths, and second power of cycles.

Theorem 1. $P_n \odot P_m$ is permuted cordial for all $n \geq 1$ and $m \geq 1$.

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 0$), and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 2$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for P_n as given in Table 2. For a given value of j with $0 \leq i'$, $j \leq 2$, we may use one of the labeling in the set $\{B_i, B_i', B''_i, C_i, C_i', C''_i\}$ for P_m, where B_i, B_i', B''_i, C_i, C_i' and C''_i are the labeling of P_m which are connected to the vertices labeled I in P_n, while B_i, B_i', B''_i, C_i, C_i' and C''_i are the labeling of P_m which are connected to the vertices labeled f in P_m as given in Table 3. Using Table 4 and the Eq. (1) and Eq. (2), we can compute the values shown in the last two columns of Table 4. Since all of these values are 1 or 0, the theorem follows.

Table 2. Labeling of P_n

$n = 3r + i$	P_n	x_i	x_f	x_g	a_i	a_f	a_g
$i = 0$	$A_0 = D_{3r}$	r	r	r	r	r	$r-1$
$i = 1$	$A_1 = D_{3r}g$	r	r	$r+1$	r	r	r
$i = 1$	$A'_1 = W_{3r}f$	r	$r+1$	r	r	r	r
$i = 1$	$A''_1 = M_{3r}i$	$r+1$	r	r	r	r	r
$i = 2$	$A_2 = E_{3r}gi$	$r+1$	r	$r+1$	r	r	$r+1$
$i = 2$	$A'_2 = S_{3r}gf$	$r+1$	r	$r+1$	$r+1$	r	r
$i = 2$	$A''_2 = D_{3r}ig$	$r+1$	r	$r+1$	$r+1$	r	r

Table 3. Labeling of P_m

$m = 3k + j$	P_m	y_i	y_f	y_g	b_i	b_f	b_g
$i = 0$	$B_0 = M_{3k}$	k	k	k	k	$k-1$	k
$i = 0$	$B'_0 = D_{3k}$	k	k	k	$k-1$	k	k
$i = 0$	$B''_0 = W_{3k}$	k	k	k	k	k	$k-1$
$i = 0$	$C_0 = D_{3k}$	k	k	k	$k-1$	k	k
$i = 0$	$C'_0 = M_{3k}$	k	k	k	k	$k-1$	k
$i = 0$	$C''_0 = W_{3k}$	k	k	k	k	k	$k-1$
$i = 1$	$B_1 = M_{3k}i$	$k+1$	k	k	k	k	k
$i = 1$	$B'_1 = P_{3k}f$	k	$k+1$	k	k	k	k
$i = 1$	$B''_1 = W_{3k}g$	k	k	$k+1$	k	k	k
$i = 1$	$C_1 = E_{3k}f$	k	$k+1$	k	k	k	k
$i = 1$	$C'_1 = D_{3k}g$	k	k	$k+1$	k	k	k
$i = 1$	$C''_1 = S_{3k}i$	$k+1$	k	k	k	k	k
$i = 2$	$B_2 = M_{3k}fi$	$k+1$	$k+1$	k	k	k	$k+1$
$i = 2$	$B'_2 = S_{3k}gi$	$k+1$	k	$k+1$	k	$k+1$	k
$i = 2$	$B''_2 = S_{3k}gi$	$k+1$	k	$k+1$	k	$k+1$	k
$i = 2$	$C_2 = W_{3k}gf$	k	$k+1$	$k+1$	$k+1$	k	k
$i = 2$	$C'_2 = D_{3k}gi$	$k+1$	k	$k+1$	k	k	$k+1$
$i = 2$	$C''_2 = M_{3k}if$	$k+1$	$k+1$	k	k	$k+1$	k
$i = 2$	$D_2 = W_{3k}gf$	$k+1$	$k+1$	k	k	k	$k+1$
$i = 2$	$D'_2 = D_{3k}gi$	k	$k+1$	$k+1$	$k+1$	k	k
$i = 2$	$D''_2 = M_{3k}if$	$k+1$	k	$k+1$	k	$k+1$	k

Table 4. Labeling of $P_n \odot P_m$

$n = 3r + i$	$m = 3r + j$	P_n	P_m	$\lvert v(x) - v(y)\rvert$, $x \neq y$, $x, y \in \{i, f, g\}$	$\lvert e(x) - e(y)\rvert$ $x \neq y$, $x, y \in \{i, f, g\}$
$i = 0$	$j = 0$	A_0	$B''_0, B'_0, B_0, \ldots$	0, 0, 0	0, 0, 0
$i = 0$	$j = 1$	A_0	$B''_1, B'_1, B_1, \ldots$	1, 1, 0	1, 1, 0
$i = 0$	$j = 2$	A_0	$B''_2, B'_2, B_2, \ldots$	1, 1, 0	1, 1, 0
$i = 1$	$j = 0$	A_1	$B''_0, B'_0, B_0, \ldots, B''_0$	0, 0, 0	0, 0, 0
$i = 1$	$j = 1$	A'_1	$C'_1, C''_1, C_1, \ldots, C'_1$	1, 1, 0	1, 1, 0
$i = 1$	$j = 2$	A''_1	$C_2, C''_2, C'_2, \ldots, C_2$	1, 1, 0	1, 1, 0
$i = 2$	$j = 0$	A_2	$C''_0, C_0, C'_0, \ldots, C''_0, C_0$	0, 0, 0	0, 0, 0
$i = 2$	$j = 1$	A'_2	$C'_1, C_1, C''_1, \ldots, C''_1 C'_1$	1, 1, 0	0, 0, 0
$i = 2$	$j = 2$	A''_2	$D''_2, D'_2, D_2, \ldots, D_2, D''_2$	1, 1, 0	1, 1, 0

Theorem 2. $C_n \odot C_m$ is permuted cordial for all $n \geq 3$ and $m \geq 3$ except at $n = 2(mode3)$ with $m = 2(mode3)$.

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 1$), and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 1$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for C_n as given in Table 5. For a given value of j with $0 \leq i'$, $j \leq 2$, we may use one of the labeling in the set $\{B_i, B_i', B''_i, C_i, C_i', C''_i\}$ for C_m, where B_i, B_i', B''_i, C_i, C_i' and C''_i are the labeling of C_m which are connected to the vertices labeled I in C_n, while B_i, B_i', B''_i, C_i, C_i' and C''_i are the labeling of C_m which are connected to the vertices labeled f in C_m as given in Table 6. Using Table 7 and Eq. (1) and Eq. (2), we can compute the values shown in the last two columns of Table 7. Since all of these values are 1 or 0, the theorem follows.

Table 5. Labeling of C_n

$n = 3r + i$	C_n	x_i	x_f	x_g	a_i	a_f	a_g
$i = 0$	$A_0 = W_{3r}$	r	r	r	r	r	r
$i = 1$	$A_1 = P_{3r}i$	$r+1$	r	r	$r+1$	r	r
$i = 2$	$A_2 = P_{3r}gf$	r	$r+1$	$r+1$	r	$r+1$	$r+1$
$i = 2$	$A'_2 = W_{3r}ig$	$r+1$	r	$r+1$	$r+1$	$r+1$	r
$i = 2$	$A"_2 = D_{3r}gi$	$r+1$	r	$r+1$	r	r	$r+2$

Table 6. Labeling of C_m

$m = 3k + j$	C_m	y_i	y_f	y_g	b_i	b_f	b_g
i = 0	$B_0 = W_{3k}$	k	k	k	k	k	k
$i = 0$	$B'_0 = W_{3k}$	k	k	k	k	k	k
$i = 0$	$B''_0 = W_{3k}$	k	k	k	k	k	k
$i = 1$	$B_1 = iP_{3k}$	$k+1$	k	k	$k-1$	$k+1$	$k+1$
$i = 1$	$B'_1 = gW_{3k}$	k	k	$k+1$	k	$k+1$	k
$i = 1$	$B''_1 = fP_{3k}$	k	$k+1$	k	k	k	$k+1$
$i = 1$	$C_1 = fP_{3k}$	k	$k+1$	k	k	k	$k+1$
$i = 1$	$C'_1 = gW_{3k}$	k	k	$k+1$	k	$k+1$	k
$i = 1$	$C''_1 = iP_{3k}$	$k+1$	k	k	$k+$	k	k
$i = 2$	$B_2 = ifE_{3k}$	$k+1$	k	$k+1$	$k+1$	$k+1$	k
$i = 2$	$B'_2 = fgD_{3k}$	k	$k+1$	$k+1$	k	$k+1$	$k+1$
$i = 2$	$B''_2 = igW_{3k}$	$k+1$	$k+1$	k	$k+1$	k	$k+1$
$i = 2$	$C_2 = P_{3k}gf$	k	$k+1$	$k+1$	k	$k+1$	$k+1$
$i = 2$	$C'_2 = gW_{3k}i$	$k+1$	k	$k+1$	$k+1$	$k+1$	k
$i = 2$	$C"_2 = fE_{3k}i$	k + 1	k + 1	k	k + 1	k	k + 1

Table 7. Labeling of $C_n \odot C_m$

$n = 3r + i$	$m = 3r + j$	C_n	C_m	$\lvert v(x) - v(y)\rvert$ $x \neq y$, $x, y \in \{i, f, g\}$	$\lvert e(x) - e(y)\rvert$ $x \neq y$, $x, y \in \{i, f, g\}$
$i = 0$	$j = 0$	A_0	$B''_0, B'_0, B_0, \ldots$	0, 0, 0	0, 0, 0
$i = 0$	$j = 1$	A_0	$B''_1, B'_1, B_1, \ldots$	1, 1, 0	1, 1, 0
$i = 0$	$j = 2$	A_0	$B''_2, B'_2, B_2, \ldots$	1, 1, 0	1, 1, 0
$i = 1$	$j = 0$	A_1	$B_0, B'_0, B''_0, .., B_0$	0, 0, 0	0, 0, 0
$i = 1$	$j = 1$	A_1	$C_1, C'_1, C''_1, .., C_1$	1, 1, 0	1, 1, 0
$i = 1$	$j = 2$	A_1	$C_2, C'_2, C''_2, .., C_2$	1, 1, 0	1, 1, 0
$i = 2$	$j = 0$	A_2	$B'_0, B_0, B''_0, \ldots, B''_0, B_{\prime 0}$	0, 0, 0	0, 0, 0
$i = 2$	$j = 1$	A'_2	$B''_1, B'_1, B_1, \ldots, B_1, B''_1$	1, 1, 0	1, 1, 0

Theorem 3. $P_n \odot C_m$ is permuted cordial for all $n \geq 1$ and $m \geq 3$.

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 0$), and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 1$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for P_n as given in Table 8. For a given value of j with $0 \leq i'$, $j \leq 2$, we may use one of the labeling in the set $\{B_i, B'_i, B''_i, C_i, C'_i, C''_i\}$ for C_m, where B_i, B'_i, B''_i, C_i, C'_i and C''_i are the labeling of C_m which are connected to the vertices labeled I in P_n, while B_i, B'_i, B''_i, C_i, C'_i and C''_i are the labeling of C_m which are connected to the vertices labeled f in C_m as given in Table 9. Using Table 10 and Eq. (1) and Eq. (2), we can compute the values shown in the last two columns of Table 10. Since all of these values are 1 or 0, the theorem follows.

Table 8. Labeling of P_n

$n = 3r + i$	P_n	x_i	x_f	x_g	a_i	a_f	a_g
$i = 0$	$A_0 = D_{3r}$	r	r	r	r	r	$r - 1$
$i = 1$	$A_1 = D_{3r}g$	r	r	$r + 1$	r	r	r
$i = 1$	$A'_1 = fM_{3r}$	r	$r + 1$	r	r	r	r
$i = 1$	$A''_1 = iD_{3r}$	$r + 1$	r	r	r	r	r
$i = 2$	$A_2 = D_{3r}gf$	r	$r + 1$	$r + 1$	$r + 1$	r	r
$i = 2$	$A'_2 = D_{3r}ig$	$r + 1$	r	$r + 1$	$r + 1$	r	r
$i = 2$	$A"_2 = D_{3r}gi$	r + 1	r	r + 1	r	r	r + 1

Table 9. Labeling of C_m

$m = 3k + j$	C_m	y_i	y_f	y_g	b_i	b_f	b_g
$i = 0$	$B_0 = W_{3k}$	k	k	k	k	k	k
$i = 1$	$B_1 = fE_{3k}$	k	$k + 1$	k	k	k	$k + 1$
$i = 1$	$B'_1 = E_{3k}i$	$k + 1$	k	k	$k + 1$	k	k
$i = 1$	$B''_1 = M_{3k}g$	k	k	$k + 1$	$k + 1$	$k - 1$	$k + 1$
$i = 1$	$C_1 = iP_{3k}$	$k + 1$	k	k	$k - 1$	$k + 1$	$k + 1$
$i = 1$	$C'_1 = W_{3k}g$	k	k	$k + 1$	k	$k + 1$	k
$i = 1$	$C''_1 = iP_{3k}$	k	$k + 1$	k	k	k	$k + 1$
$i = 2$	$B_2 = ifE_{3k}$	$k + 1$	$k + 1$	k	$k + 1$	k	$k + 1$
$i = 2$	$B'_2 = fgD_{3k}$	k	$k + 1$	$k + 1$	k	$k + 1$	$k + 1$
$i = 2$	$B''_2 = igW_{3k}$	$k + 1$	k	$k + 1$	$k + 1$	$k + 1$	k
$i = 2$	$C_2 = fgP_{3k}$	$k + 1$	k	$k + 1$	k	$k + 1$	$k + 1$
$i = 2$	$C'_2 = igW_{3k}$	$k + 1$	$k + 1$	k	$k + 1$	k	$k + 1$
$i = 2$	$C''_2 = ifE_{3k}$	k	$k + 1$	$k + 1$	$k + 1$	k	$k + 1$

Table 10. Labeling of $P_n \odot C_m$

$n = 3r + i$	$m = 3k + j$	P_n	C_m	$\lvert v(x) - v(y)\rvert$ $x \neq y$, $x, y \in \{i, f, g\}$	$\lvert e(x) - e(y)\rvert$ $x \neq y$, $x, y \in \{i, f, g\}$
i = 0	j = 0	A_0	$B_0, B_0, B_0, \dots$	0, 0, 0	0, 0, 0
$i = 0$	$j = 1$	A_0	$B''_1, B'_1, B_1, \dots$	1, 1, 0	1, 1, 0
$i = 0$	$j = 2$	A_0	$B''_2, B'_2, B_2, \dots$	1, 1, 0	1, 1, 0
$i = 1$	$j = 0$	A_1	$B_0, B_0, B_0, \dots, B_0$	0, 0, 0	0, 0, 0
$i = 1$	$j = 1$	A'_1	C'_1, C_1, C''_1, C'_1	1, 1, 0	1, 1, 0
$i = 1$	$j = 2$	A''_1	$C_2, C''_2, C'_2, \dots, C_2$	1, 1, 0	1, 1, 0
$i = 2$	$j = 0$	A_2	$B_0, B_0, B_0, \dots, B_0, B_0$	0, 0, 0	0, 0, 0
$i = 2$	$j = 1$	A'_2	$C''_1, C'_1, C_1, \dots, C_1, C''_1$	1, 1, 0	1, 1, 0
$i = 2$	$j = 2$	A''_2	$C''_2, C'_2, C_2, \dots, C''_2, C_2$	1, 1, 0	1, 1, 0

Theorem 4. $C_n \odot P_m$ is permuted cordial for all $n \geq 3$ and $m \geq 1$.

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 1$), and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 0$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for C_n as given in Table 11. For a given value of j with $0 \leq i'$, $j \leq 2$, we may use one of the labeling in the set $\{B_i, B'_i, B''_i, C_i, C'_i, C''_i\}$ for C_m, where B_i, B'_i, B''_i, C_i, C'_i and C''_i are the labeling of P_m which are connected to the vertices labeled I in C_n, while B_i, B'_i, B''_i, C_i, C'_i and C''_i are the labeling of P_m which are connected to the vertices labeled f in P_m as given in Table 12. Using Table 13 and Eq. (1) and Eq.(2), we can compute the values shown in the last two columns of Table 13. Since all of these values are 1 or 0, the theorem follows.

Table 11. Labeling of C_n

$n = 3r + i$	C_n	x_i	x_f	x_g	a_i	a_f	a_g
i = 0	$A_0 = D_{3r}$	r	r	r	r	r	r
$i = 1$	$A_1 = E_{3r}g$	r	r	$r + 1$	r	$r + 1$	r
$i = 1$	$A'_1 = D_{3r}g$	r	r	$r + 1$	r	$r + 1$	r
$i = 2$	$A_2 = P_{3r}gf$	r	$r + 1$	$r + 1$	$r + 1$	$r + 2$	$r - 1$
$i = 2$	$A'_2 = D_{3r}fg$	$r + 1$	r	r + 1	r + 1	r + 1	r

Table 12. Labeling of P_m

$m = 3k + j$	P_m	y_i	y_f	y_g	b_i	b_f	b_g
$i = 0$	$B_0 = D_{3k}$	k	k	k	$k - 1$	k	k
$i = 0$	$B'_0 = M_{3k}$	k	k	k	k	$k - 1$	k
$i = 0$	$B''_0 = W_{3k}$	k	k	k	k	k	$k - 1$
$i = 0$	$C_0 = E_{3k}$	k	k	k	k	$k - 1$	k
$i = 0$	$C'_0 = M_{3k}$	k	k	k	k	k	$k - 1$
$i = 0$	$C''_0 = D_{3k}$	k	k	k	$k - 1$	k	k
$i = 0$	$D_0 = M_{3k}$	k	k	k	k	$k - 1$	k
$i = 0$	$D'_0 = D_{3k}$	k	k	k	$k - 1$	k	k
$i = 0$	$D''_0 = W_{3k}$	k	k	k	k	k	$k - 1$
$i = 1$	$B_1 = M_{3k}i$	$k + 1$	k	k	k	k	k
$i = 1$	$B'_1 = E_{3k}f$	k	$k + 1$	k	k	k	k
$i = 1$	$B''_1 = W_{3kg}$	k	k	$k + 1$	k	k	k
$i = 1$	$C_1 = P_{3k}f$	k	$k + 1$	k	k	k	k
$i = 1$	$C'_1 = W_{3k}g$	k	k	$k + 1$	k	k	k
$i = 1$	$C''_1 = S_{3k}i$	$k + 1$	k	k	k	k	k
$i = 1$	$D_1 = W_{3k}g$	k	k	$k + 1$	k	k	k
$i = 1$	$D'_1 = P_{3k}f$	k	$k + 1$	k	k	k	k
$i = 1$	$D''_1 = iS_{3k}$	$k + 1$	k	k	$k + 1$	$k + 1$	$k - 1$
$i = 2$	$B_2 = P_{3k}if$	$k + 1$	$k + 1$	k	$k - 1$	$k + 1$	$k + 1$
$i = 2$	$B'_2 = M_{3k}ig$	$k + 1$	k	$k + 1$	k	k	$k + 1$
$i = 2$	$B''_2 = E_{3k}gf$	k	$k + 1$	$k + 1$	$k + 1$	$k - 1$	$k + 1$

$i=2$	$C_2 = igW_{3k}$	$k+1$	k	$k+1$	k	$k+1$	k
$i=2$	$C'_2 = P_{3k}fg$	k	$k+1$	$k+1$	$k+1$	k	k
$i=2$	$C''_2 = E_{3k}fi$	$k+1$	$k+1$	k	k	$k+1$	k
$i=2$	$D_2 = M_{3k}fi$	$k+1$	$k+1$	k	k	k	$k+1$
$i=2$	$D'_2 = S_{3k}ig$	$k+1$	k	$k+1$	k	k	$k+1$
$i=2$	$D''_2 = D_{3k}fg$	k	$k+1$	$k+1$	k	k	$k+1$

Table 13. Labeling of $C_n \odot P_m$

$n=3r+i$	$m=3r+j$	C_n	P_m	$\lvert v(x)-v(y)\rvert$ $x \neq y$, $x,y \in \{i,f,g\}$	$\lvert e(x)-e(y)\rvert$ $x \neq y$, $x,y \in \{i,f,g\}$
$i=0$	$j=0$	A_0	$B''_0, B'_0, B_0, \ldots$	0, 0, 0	0, 0, 0
$i=0$	$j=1$	A_0	$B''_1, B'_1, B_1, \ldots$	1, 1, 0	1, 1, 0
$i=0$	$j=2$	A_0	$B''_2, B'_2, B_2, \ldots$	1, 1, 0	1, 1, 0
$i=1$	$j=0$	A_1	$D''_0, D_0, D'_0, \ldots, D''_0$	0, 0, 0	0, 0, 0
$i=1$	$j=1$	A'_1	$C''_1, C'_1, C_1, \ldots, C''_1$	1, 1, 0	1, 1, 0
$i=1$	$j=2$	A'_1	$C''_2, C'_2, C_2, \ldots, C''_2$	1, 1, 0	1, 1, 0
$i=2$	$j=0$	A_2	$D_0, D'_0, D''_0, \ldots, D''_0 D'_0$	0, 0, 0	0, 0, 0
$i=2$	$j=1$	A_2	$D''_1, D'_1, D_1, \ldots, D'_1, D''_1$	1, 1, 0	1, 1, 0
$i=2$	$j=2$	A'_2	$D_2, D'_2, D''_2, \ldots, D'_2 D_2$	1, 1, 0	1, 1, 0

Theorem 5. $P_n \odot P_m^2$ is permuted cordial for all $n \geq 1$ and $m \geq 1$.

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 0$), and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 2$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for P_n as given in Table 14. For a given value of j with $0 \leq i'$, $j \leq 2$, we may use one of the labeling in the set $\{B_i, B_i', B''_i, C_i, C_i', C''_i\}$ for P_m^2, where B_i, B_i', B''_i, C_i, C_i' and C''_i are the labeling of P_m^2 which are connected to the vertices labeled I in P_n, while B_i, B_i', B''_i, C_i, C_i' and C''_i are the labeling of P_m^2 which are connected to the vertices labeled f in P_m^2 as given in Table 15. Using Table 16 and Eq. (1) and Eq. (2), we can compute the values shown in the last two columns of Table 16. Since all of these values are 1 or 0, the theorem follows.

Table 14. Labeling of P_n

$n=3r+i$	P_n	x_i	x_f	x_g	a_i	a_f	a_g
$i=0$	$A_0 = D_{3r}$	r	r	r	r	r	$r-1$
$i=1$	$A_1 = D_{3r}g$	r	r	$r+1$	r	r	r
$i=2$	$A_2 = D_{3r}gf$	r	$r+1$	$r+1$	$r+1$	r	r
$i=2$	$A'_2 = fgE_{3r}$	r	$r+1$	$r+1$	r	$r+1$	r
$i=2$	$A''_2 = ifW_{3r}$	$r+1$	$r+1$	r	r	r	r + 1

Table 15. Labeling of $P^2{}_m$

$m=3k+j$	$P^2{}_m$	y_i	y_f	y_g	b_i	b_f	b_g
$i=0$	$B_0 = E_{3k}$	k	k	k	$2k-1$	$2k-1$	$2k-1$
$i=0$	$B'_0 = P_{3k}$	k	k	k	$2k-1$	$2k-1$	$2k-1$
$i=0$	$B''_0 = W_{3k}$	k	k	k	$2k-1$	$2k-1$	$2k-1$
$i=1$	$B_1 = M_{3k}f$	k	$k+1$	k	$2k$	$2k-1$	$2k$
$i=1$	$B'_1 = M_{3k}g$	k	k	$k+1$	$2k$	$2k$	$2k-1$
$i=1$	$B''_1 = iE_{3k}$	$k+1$	k	k	$2k-1$	$2k$	$2k$
$i=1$	$C_1 = M_{3k}f$	k	$k+1$	k	$2k$	$2k-1$	$2k$
$i=1$	$C'_1 = iE_{3k}$	k	k	$k+1$	$2k-1$	$2k$	$2k$
$i=1$	$C''_1 = iS_{3k}$	$k+1$	k	k	$2k$	$2k$	$2k-1$
$i=1$	$D_1 = M_{3k}g$	k	k	$k+1$	$2k$	$2k$	$2k-1$
$i=1$	$D'_1 = P_{3k}f$	k	$k+1$	k	$2k$	$2k$	$2k-1$
$i=1$	$D''_1 = iS_{3k}$	$k+1$	k	k	$2k$	$2k$	$2k-1$
$i=2$	$B_2 = D_{3k}fg$	k	$k+1$	$k+1$	$2k+1$	$2k$	$2k$
$i=2$	$B'_2 = igE_{3k}$	$k+1$	k	$k+1$	$2k$	$2k+1$	$2k$

$i = 2$	$B''_2 = ifW_{3k}$	$k+1$	$k+1$	k	$2k$	$2k$	$2k+1$
$i = 2$	$C_2 = D_{3k}fg$	k	$k+1$	$k+1$	$2k+1$	$2k$	$2k$
$i = 2$	$C'_2 = M_{3k}ig$	$k+1$	k	$k+1$	$2k$	$2k$	$2k+1$
$i = 2$	$C''_2 = M_{3k}fi$	$k+1$	$k+1$	k	$2k$	$2k+1$	$2k$
$i = 2$	$D_2 = E_{3k}gf$	$k+1$	k	$k+1$	$2k$	$2k+1$	$2k$
$i = 2$	$D'_2 = M_{3k}fi$	k	$k+1$	$k+1$	$2k$	$2k+1$	$2k$
$i = 2$	$D''_2 = igE_{3k}$	$k+1$	$k+1$	k	$2k$	$2k+1$	$2k$

Table 16. Labeling of $P_n \odot P^2{}_m$

$n = 3r + i$	$m = 3r + j$	P_n	$P^2{}_m$	$\lvert v(x) - v(y)\rvert$ $x \neq y$, $x, y \in \{i, f, g\}$	$\lvert e(x) - e(y)\rvert$ $x \neq y$, $x, y \in \{i, f, g\}$
$i = 0$	$j = 0$	A_0	$B''_0, B'_0, B_0, \ldots$	0, 0, 0	0, 0, 0
$i = 0$	$j = 1$	A_0	$B''_1, B'_1, B_1, \ldots$	1, 1, 0	1, 1, 0
$i = 0$	$j = 2$	A_0	$B''_2, B'_2, B_2, \ldots$	1, 1, 0	1, 1, 0
$i = 1$	$j = 0$	A_1	$B''_0, B'_0, B_0, \ldots, B''_0$	0, 0, 0	0, 0, 0
$i = 1$	$j = 1$	A_1	$C''_1, C'_1, C_1, \ldots, C''_1$	1, 1, 0	1, 1, 0
$i = 1$	$j = 2$	A_1	$C''_2, C'_2, C_2, \ldots, C''_2$	1, 1, 0	1, 1, 0
$i = 2$	$j = 0$	A_2	$B''_0, B'_0, B_0, \ldots, B''_0, B'_0$	0, 0, 0	0, 0, 0
$i = 2$	$j = 1$	A'_2	$D'_1, D''_1, D''_1, D_1, D'_1, \ldots$	1, 1, 0	1, 1, 0
$i = 2$	$j = 2$	A''_2	$D_2, D'_2, D'_2, D''_2, D_2, \ldots$	1, 1, 0	1, 1, 0

Theorem 6. $P_n \odot C_m^2$ is permuted cordial for all $n \geq 1$ and $m \geq 3$.

Proof. Let $n = 3r + i'$ ($i' = 0,1,2$ and $r \geq 0$), and $m = 3k + j$ ($j = 0,1,2$ and $k \geq 1$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for P_n as given in Table 17. For a given value of j with $0 \leq i'$, $j \leq 2$, we may use one of the labeling in the set $\{B_i, B'_i, B''_i, C_i, C'_i, C''_i\}$ for C_m^2, where B_i, B'_i, B''_i, C_i, C'_i and C''_i are the labeling of C_m^2 which are connected to the vertices labeled I in P_n, while B_i, B'_i, B''_i, C_i, C'_i and C''_i are the labeling of C_m^2 which are connected to the vertices labeled f in C_m^2 as given in Table 18. Using Table 19 and Eq. (1) and Eq. (2), we can compute the values shown in the last two columns of Table 19. Since all of these values are 1 or 0, the theorem follows.

Table 17. Labeling of P_n

$n = 3r + i$	P_n	x_i	x_f	x_g	a_i	a_f	a_g
$i = 0$	$A_0 = D_{3r}$	r	r	r	r	r	$r-1$
$i = 1$	$A_1 = D_{3r}g$	r	r	$r+1$	r	r	r
$i = 2$	$A_2 = P_{3r}gf$	r	$r+1$	$r+1$	r	$r+1$	r

Table 18. Labeling of $C^2{}_m$

$m = 3k + j$	$C^2{}_m$	y_i	y_f	y_g	b_i	b_f	b_g
$i = 0$	$B_0 = E_{3k}$	k	k	k	$2k-1$	$2k$	$2k-1$
$i = 0$	$B'_0 = P_{3k}$	k	k	k	$2k$	$2k-1$	$2k-1$
$i = 0$	$B''_0 = W_{3k}$	k	k	k	$2k-1$	$2k-1$	$2k$
$i = 0$	$C_0 = D_{3k}$	k	k	k	$2k$	$2k-1$	$2k-1$
$i = 0$	$C'_0 = E_{3k}$	k	k	k	$2k-1$	$2k$	$2k-1$
$i = 0$	$C''_0 = S_{3k}$	k	k	k	$2k-1$	$2k-1$	$2k$
$i = 1$	$B_1 = W_{3k}f$	k	$k+1$	k	$2k$	$2k$	$2k$
$i = 1$	$B'_1 = E_{3k}g$	k	k	$k+1$	$2k$	$2k$	$2k$
$i = 1$	$B''_1 = S_{3k}i$	$k+1$	k	k	$2k$	$2k$	$2k$
$i = 2$	$B_2 = S_{3k}fi$	k	$k+1$	$k+1$	$2k+1$	$2k+1$	$2k$
$i = 2$	$B'_2 = S_{3k}gi$	$k+1$	k	$k+1$	$2k+1$	$2k$	$2k+1$
$i = 2$	$B''_2 = D_{3k}fg$	$k+1$	$k+1$	k	$2k$	$2k+1$	$2k+1$
$i = 2$	$C_2 = D_{3k}fg$	k	$k+1$	$k+1$	$2k+1$	$2k$	$2k$
$i = 2$	$C'_2 = S_{3k}gi$	$k+1$	k	$k+1$	$2k+1$	$2k$	$2k+1$
$i = 2$	$C''_2 = S_{3k}fi$	$k+1$	$k+1$	k	$2k$	$2k+1$	$2k+1$

$i=2$	$D_2 = S_{3k}gi$	k	$k+1$	$k+1$	$2k+1$	$2k+1$	$2k$
$i=2$	$D'_2 = S_{3k}fi$	$k+1$	k	$k+1$	$2k+1$	$2k$	$2k+1$
$i=2$	$D''_2 = D_{3k}fg$	$k+1$	$k+1$	k	$2k$	$2k+1$	$2k+1$

Table 19. Labeling of $P_n \odot C^2{}_m$

$n=3r+i$	$m=3r+j$	P_n	$C^2{}_m$	$\lvert v(x)-v(y)\rvert$ $x \neq y$, $x,y \in \{i,f,g\}$	$\lvert e(x)-e(y)\rvert$ $x \neq y$, $x,y \in \{i,f,g\}$
$i=0$	$j=0$	A_0	B''_0, B'_0, B_0	0, 0, 0	0, 0, 0
$i=0$	$j=1$	A_0	$B''_1, B'_1, B_1, \ldots$	1, 1, 0	1, 1, 0
$i=0$	$j=2$	A_0	$B''_2, B'_2, B_2, \ldots$	1, 1, 0	1, 1, 0
$i=1$	$j=0$	A_1	$C''_0, C'_0, C_0, .., C''_0$	0, 0, 0	0, 0, 0
$i=1$	$j=1$	A_1	$B''_1, B'_1, B_1, .., B''_1$	1, 1, 0	1, 1, 0
$i=1$	$j=2$	A_1	$B''_2, B'_2, B_2, .., B''_2$	1, 1, 0	1, 1, 0
$i=2$	$j=0$	A_2	$B'_0, B_0, B''_0, \ldots, B''_0, B'_0$	0, 0, 0	0, 0, 0
$i=2$	$j=1$	A_2	$B'_1, B_1, B''_1, \ldots, B''_1, B'_1$	1, 1, 0	1, 1, 0
$i=2$	$j=2$	A_2	$B'_2, B_2, B''_2, \ldots, B''_2, B'_2$	1, 1, 0	1, 1, 0

4. Algorithm

In this section, we propose an algorithm for calculating the permuted cordial labeling for any graph G. This algorithm provides a framework for attempting to find a permuted cordial labeling for a given graph. It's important to note that not all graphs will admit a permuted cordial labeling, and the specific method of assigning labels to vertices in step 3 can vary based on the graph's structure and properties. we assume that labeling each vertex and edge takes constant time, then the time complexity of Algorithm 1 is $O(n)$, since each vertex and edge is visited once.

Table 20. Algorithm for Permuted cordial labeling of a graph

Algorithm1: Permuted Cordial Labeling of a Graph
Input: A graph $G(V,E)$
Output: A Permuted cordial labeling of G if it exists

Begin
Step 1. Initialize three counters, $v(I)$, $v(f)$ and $v(g)$, to 0.
Step 2. Initialize three counters, $e(I)$, $e(f)$ and $e(g)$, to 0.
Step 3. For each vertex v in V:
a. Assign a label $f(v) \in \{I, f, g\}$ to the vertex v.
b. If $f(v) == I$, increment $v(I)$; else If $f(v) == f$, increment $v(f)$; else increment $v(g)$.
Step 4. For each edge $e(u,v)$ in E:
a. Assign a label $f(e) = f(u)$ o $f(v)$ to the edge e.
b. If $f(e) == I$, increment $e(I)$; If $f(e) == f$, increment $e(f)$; else increment $e(g)$.
Step 5. Check the Permuted cordial condition:
a. If $|v(x) - v(y)| \leq 1$ and $|e(x) - e(y)| \leq 1$, $x \neq y$ and $x, y \in \{i, f, g\}$:
i. The labeling is cordial.
ii. Output the labeling of vertices and edges.
b. Else:
i. The graph G does not admit a Permuted cordial labeling.
ii. Output that no Permuted cordial labeling exists.
End Algorithm

5. Discussion

In graph theory, permuted cordial labeling is an intriguing idea with possible applications in a number of industrial domains. A graph's vertices are given labels using this labeling technique so that the total of the labels of nearby vertices satisfies predetermined requirements. In engineering applications, this idea can be useful in the following ways:

5.1. Network design and optimization

In telecommunications and computer networks, it is important for the efficient routing of data. Permuted cordial labeling can be a means of optimization for network topologies that can minimize the number of connections among them while still having robust communication paths. Applying this labeling, engineers are able to design networks with load balancing and low latency which in turn results in a better performance.

5.2. Resource allocation

In systems, wherein resources, such as bandwidth or power, must be allocated, the permuted cordial labeling becomes a just and effective model of distribution. E.g. In a wireless sensor network, this labeling will be a great deal in the arrangement of the sensor nodes in the way of having maximum coverage and minimum energy consumption.

6. Conclusions

We proved that each $P_n \odot P_m$, $n, m \geq 2$ admits permuted cordial labeling. Each $C_n \odot C_m$, n, $m \geq 3$ admits permuted cordial labeling for all n, $m \neq (1mod3; 2mod3)$. Each $P_n \odot C_m$, $n \geq 1$, $m \geq 3$ admits a permuted cordial labeling. $C_n \odot P_m$, $n \geq 3$, $m \geq 1$ admits permuted cordial labeling.

Moreover, we proved that the corona of $P_n \odot P^2{}_m$, n, $m \geq 2$ admits a permuted cordial labeling. Also, we proved the corona of $P_n \odot C^2{}_m$, $n \geq 2$, $m > 4$ admits a permuted cordial labeling. We proposed an algorithm for determining the Permuted cordial of a given graph. In the future, we will apply permuted cordial labeling to other types of graphs.

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Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Author contributions

Khalid A. Alsatami: conceptualization, validation, data curation. Yasmin Algrawani: methodology, formal analysis, supervision. Atef Abd El-hay: software, investigation, resources, visualization, supervision.

Conflict of interest

The authors declare that they have no conflict of interest.

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